Master in Financial Engineering (EPFL)

Financial Econometrics

Exercises session 1: Time Series Analysis

Weakly stationary processes

Exercise 1: Characterizing second-order stationarity
Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ denoted a (weak) white noise with variance $\sigma_\epsilon^2 > 0$. Discuss about the weak stationarity of the following processes $(X_t)_{t\in\mathbb{Z}}$.

1. $X_t = \varepsilon_t - \varepsilon_{t-1}$;
2. $X_t = a + b\varepsilon_t + c\varepsilon_{t-1}$
3. For $t \geq 0$, $X_t - X_{t-1} = \varepsilon_t$ (one further assumes that $\forall t > 0, \varepsilon_t \perp\!\!\!\perp X_0$).

Exercise 2: Linear transformation of a weakly stationary process

1. Let (X_t) denote a weakly stationary stochastic process that has the following linear representation:

$$X_t = \mu + \sum_{k=0}^{\infty} \theta_k \epsilon_{t-k}$$

Show that

$$\begin{aligned}
\mathbb{E}[X_t] &= \mu \\
\mathbb{V}[X_t] &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} \theta_k^2 \\
\mathbb{C}ov[X_t, X_{t-h}] &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} \theta_k \theta_{k+h}.
\end{aligned}$$

Autocorrelation function

Exercise 3: Autocorrelation function of an autoregressive process
Consider an AR(1) process

$$X_t = \frac{4}{5} X_{t-1} + \eta_t \tag{1}$$

where η_t is a weak white noise.
Part I: ACF using the infinite moving average representation

1. Show by backward induction (assuming that $X_{-\infty}$ is bounded) that

$$X_t = \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k \eta_{t-k}$$

2. Find $\mathbb{V}\left[X_t\right]$;

3. Find the autocorrelation function

Part II: ACF using the Yule-Walker equation

1. Multiply Eq. (1) and take the expectation on both sides. What is the corresponding equation? Interpret it.

2. Show that the autocorrelation function is driven by a difference equation of order 1.

3. Solve this difference equation and compare the results with those of Part I.

Part III: PACF

1. Show the first partial autocorrelation equals the first autocorrelation.

2. One student claims that the partial autocorrelation function is only different from zero for $h = 1$. Is it correct? Explain carefully.

Part IV: Extensions

1. Consider the following AR(2) process

$$X_t = \frac{1}{3} + \frac{5}{6} X_{t-1} - \frac{1}{6} X_{t-2} + \eta_t$$

   where η_t is a weak white noise. Derive the Yule-Walker equation.

2. Write down the Yule-Walker equation in the case of an AR(p) process.

Exercise 4: Autocorrelation function of an ARMA(1,1) process
Let (X_t) denote the following stochastic process

$$X_t = \frac{1}{3} + \frac{1}{8} X_{t-1} + \epsilon_t - \frac{3}{4} \epsilon_{t-1}$$

where ϵ_t is a weak white noise $(0, \sigma_\epsilon^2)$.

1. What is the autocorrelation function of the moving average part?

2. Write down the Yule-Walker equations for $h = 0, 1$.

3. Find the Yule-Walker equation for $|h| > 2$

Best linear forecasts

Exercise 5: Forecasts with autoregressive processes
Consider the following AR(p) stochastic processes

(i) $X_t = \frac{1}{2} + 0.8X_{t-1} + \epsilon_t$;

(ii) $X_t = \frac{5}{6}X_{t-1} - \frac{1}{6}X_{t-2} + \epsilon_t$.

where ϵ_t is a weak white noise $(0, \sigma_\epsilon^2)$.

For each stochastic process:

1. Determine the best linear forecast for $h = 1, 2$.
2. Determine the forecast error and its variance $h = 1, 2$.
3. Show that the best linear linear forecast is defined by a difference equation for $h \geq p$.
4. What happens as $h \to \infty$?

Exercise 6: Forecasts with a moving average process
Let (X_t) denote the following stochastic process

$$X_t = \frac{1}{3} + \epsilon_t - \frac{3}{4}\epsilon_{t-1} + \frac{1}{8}\epsilon_{t-2}$$

where ϵ_t is a weak white noise $(0, \sigma_\epsilon^2)$.

1. Determine the best linear forecast for $h = 1, 2, 3$.
2. Determine the forecast error and its variance for $h = 1, 2, 3$.

Exercise 7: Modeling and forecasting a time series: Consider the interest rate spread variable over the period 1960Q1-2010Q1.

1. Comment Figures 1, 2, and 3. What can be say regarding the stationarity of the series?

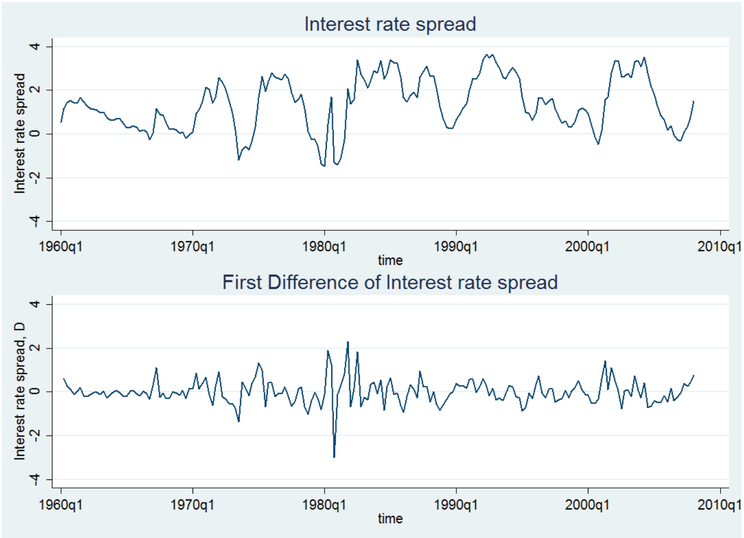

Figure 1: (First-difference of) Interest rate spread

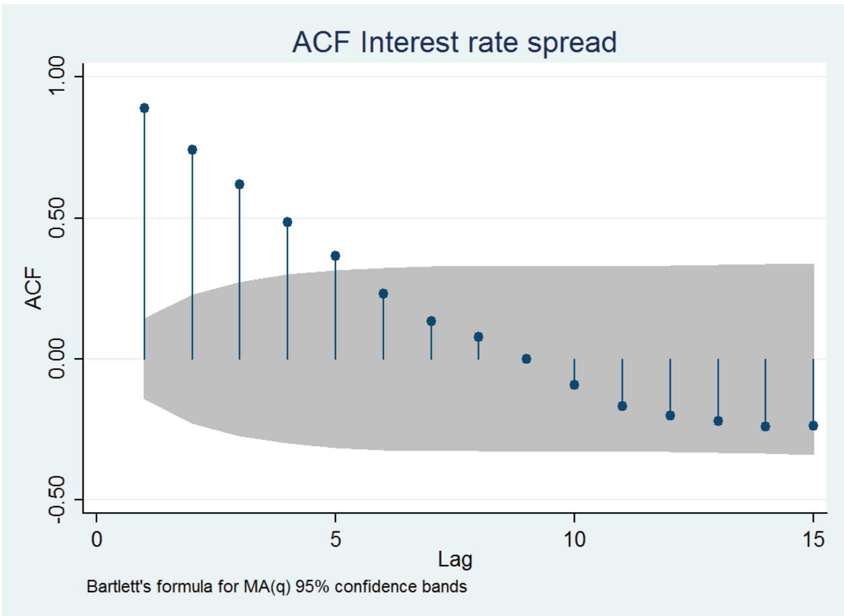

Figure 2: `ACF` of X_t

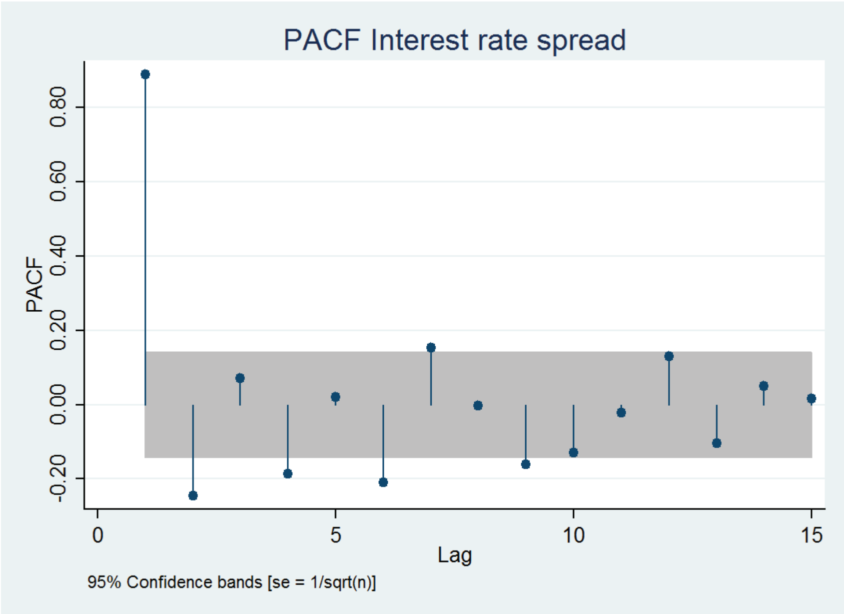

Figure 3: `ACF` of X_t

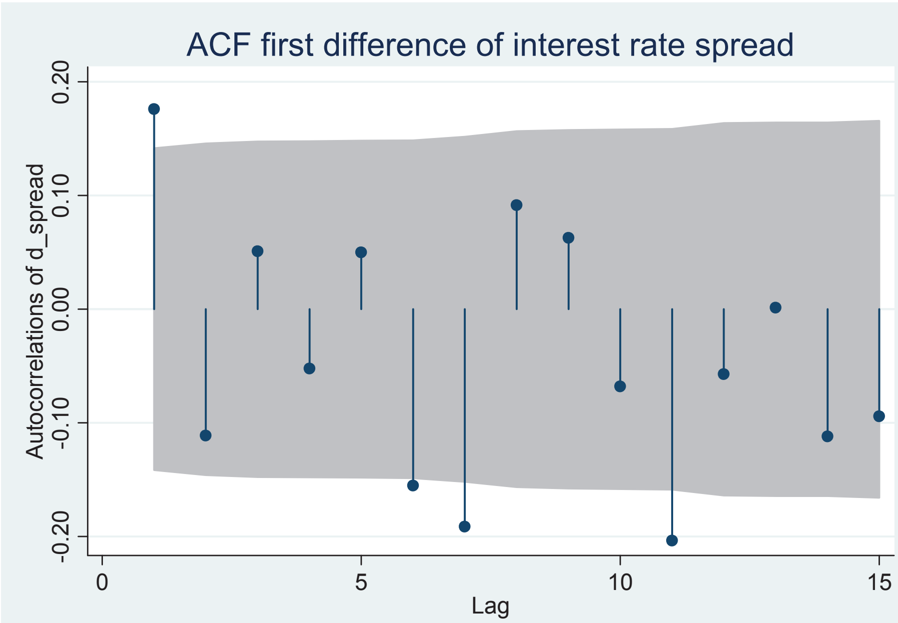

Figure 4: `ACF` of ΔX_t

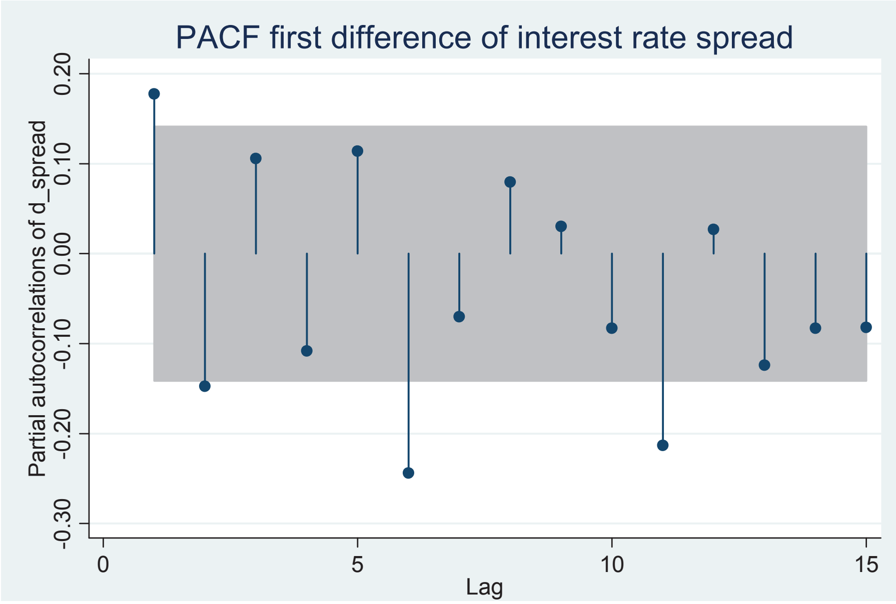

Figure 5: `PACF` of ΔX_t

2. Different unit root tests are conducted in order to identify d.

   (a) Using the first-difference test regression, the `DF` unit root test is presented in Table 1.

Table 1: `DF` unit root test

Parameter	Estimation	Std. Err.	t-stat.	p-value
ϕ	-0.1111	0.0330	-3.3624	0.0009
c	0.1575	0.0603	2.6083	0.0098

   Note: Tabulated critical values of the Student test statistic at 1%, 5% et 10% are respectively -3.464, -2.876 and -2.574,.

   - Comment the specification. Write down the null and alternative hypothesis.
   - Determine the `DF` Student test statistic.
   - Comment the results

   (b) Table 2 provides the results of the `ADF` unit root test using `SBIC` (the maximum number of lags is 12) and the first-difference test regression.

Table 2: `ADF` unit root test

Parameter	Estimation	Std. Err.	t-stat.	p-value
		A. One lag		
$\phi^*(1) = -\phi(1)$	-0.1362	0.0331	-4.1183	0.0001
α_1	0.2454	0.7079	3.4663	0.0007
c	0.1890	0.0598	3.1588	0.0018

Note: Tabulated critical values of the Student test statistic at 1%, 5% et 10% are respectively -3.464, -2.876 and -2.574,.

- Comment the results

(c) Table 3 provides the results of the PP unit root test.

Table 3: PP unit root test

Parameter	Estimation	Std. Err.	t-stat.	p-value
ϕ	-0.1111	0.0330	-3.3624	0.0009
c	0.1575	0.0603	2.6083	0.0098

Note: Tabulated critical values of the Student test statistic at 1%, 5% et 10% are respectively -3.464, -2.876 and -2.574,.

The PP test statistic is given by -3.7910.

- Why the results of Table 3 are the same as the ones of Table 1?
- Comment the results.

(d) Finally, a KPSS unit root test (with only a constant) is conducted. The LM test statistic is given by 0.2959. The asymptotic critical values are given by 0.739 (1% level), 0.463 (5% level), and 0.347 (10% level). Comment.

(e) What is you conclusion regarding the order d?

3. Using the autocorrelation function (Figure 2) and the partial autocorrelation function (Figure 3), determine some orders p and q. Explain.

4. Table 4 reports the information criteria AIC (panel A), SBIC (panel B), and HQ (panel C).

   - Comment the results for each information criterion.
   - Which model(s) do you suggest? Explain.

5. Table 5 reports the estimation of three models : ARMA(2,6), ARMA(2,7), and ARMA(2,(1,7)).[1]

   - Comment the different estimations. In particular, how can we explain the estimation results of the second specification (ARMA$(2,7)$).
   - Compare the information criteria of the last model (ARMA(2,(1,7))) with those of Table 4. What does it suggest?

6. Using the previous results, different models are estimated (Table 6). In this respect, Table 7 displays the (sample) autocorrelation function and the (sample) partial autocorrelation function. Moreover, the Portmanteau test is implemented for each model of the previous question.

   - Why is it useful to check the (sample) autocorrelation function of the residuals?

[1] The ARMA(2,(1,7)) specification is defined to be

$$X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_7 \epsilon_{t-7}.$$

- Write down the test (null and alternative hypothesis). What is the test statistic and the asymptotic distribution?
- Interpret the results of the test (the last column being the p-value).
- Which model(s) can one choose?

7. Finally, one-step ahead forecasts are implemented: the model is estimated with a recursive window. The first estimation is done over the period 1960Q4-1995Q3. Some results are reported in Figure 6 and Figure 7.[2]

   In order to compare the one-step ahead forecasts, one compute the Diebold-Mariano test using the `RMSE` (respectively, `MAE`).

   - Write the test. What is the asymptotic distribution?
   - Interpret the results of Table 8 and Table 9.
   - What can be concluded regarding the choice of the model?

8. As a final check, one computes the following regression

$$X_t = a + bX^*_{t-1}(1) + u_t$$

   for all t in the holdout period.

   - What is the interest of such a regression?
   - Comment the results of Table 10.

[2]The "dynamic forecast" corresponds to the multi-step forecasts given the information available at time 1995Q3.

Table 4: Information criteria

p/q	0	1	2	3	4	5	6	7
	Panel A. AIC							
0	3.2177	2.2454	1.9500	1.7221	1.6889	1.6451	1.5167	1.5279
1	1.6608	1.5909	1.5872	1.5721	1.5780	1.5804	1.5321	1.5402
2	1.6099	1.5754	1.5854	1.5764	1.5804	1.5931	1.5498	1.5530
3	1.6207	1.5896	1.5796	1.5837	1.5891	1.5485	1.5057	1.5168
4	1.6018	1.5899	1.5885	1.5760	1.5709	1.5639	1.5181	1.501666
5	1.6168	1.6059	1.6030	1.5913	1.5278	1.5691	1.5291	1.5299
6	1.5882	1.5845	1.5503	1.5683	1.5789	1.5833	1.5910	1.5409
7	1.5814	1.5922	1.5767	1.5849	1.5956	1.5530	1.5212	1.4904
	Panel B. SBIC							
0	3.2346	2.2793	2.0008	1.7898	1.7735	1.7466	1.6351	1.6632
1	1.6948	1.6419	1.6551	1.6569	1.6799	1.6993	1.6679	1.6929
2	1.6610	1.6435	1.6706	1.6786	1.6996	1.7293	1.7031	1.7233
3	1.6891	1.6751	1.6822	1.7034	1.7258	1.7024	1.6767	1.7048
4	1.6876	1.6929	1.7086	1.7133	1.7253	1.7355	1.7069	1.7075
5	1.7201	1.7265	1.7407	1.7462	1.7000	1.7585	1.7357	1.7537
6	1.7092	1.7228	1.7058	1.7411	1.7690	1.7907	1.8156	1.7828
7	1.7202	1.7483	1.7502	1.7757	1.8038	1.7785	1.7640	1.7505
	Panel C. HQ							
0	3.2246	2.2592	1.9706	1.7495	1.7232	1.6863	1.5646	1.5827
1	1.6746	1.6116	1.6147	1.6065	1.6193	1.6286	1.5871	1.6021
2	1.6306	1.6030	1.6199	1.6178	1.6287	1.6483	1.6119	1.6220
3	1.6484	1.6243	1.6212	1.6322	1.6445	1.6109	1.5750	1.5930
4	1.6366	1.6316	1.6372	1.6317	1.6335	1.6335	1.5946	1.5851
5	1.6587	1.6548	1.6588	1.6541	1.5976	1.6458	1.6128	1.6206
6	1.6373	1.6405	1.6133	1.6383	1.6560	1.6673	1.6820	1.6389
7	1.6377	1.6555	1.6470	1.6622	1.6800	1.6444	1.6196	1.5958

Table 5: Constrained and unconstrained estimation

Parameter	Estimation	Std. Err.	t-stat.	p-value
	A. ARMA(2,6)			
c	1.3869	0.2429	5.7080	0
ϕ_1	-0.0930	0.1677	-0.5544	0.5799
ϕ_2	0.3391	0.1657	2.0463	0.0422
θ_1	1.3006	0.1616	8.0462	0.0000
θ_2	0.7525	0.2505	3.0039	0.0030
θ_3	0.6522	0.2415	2.7000	0.0076
θ_4	0.4892	0.2017	2.4257	0.0163
θ_5	0.4131	0.1653	2.4985	0.0134
θ_6	0.3230	0.0858	3.7625	0.0002
AIC	1.5498	SBIC	1.7030	
	B. ARMA(2,7)			
c	1.3957	0.2603	5.3599	0.0000
ϕ_1	0.6093	0.6053	1.0067	0.3154
ϕ_2	0.1221	0.2276	0.5367	0.5921
θ_1	0.5965	0.6029	0.9894	0.3238
θ_2	0.1418	0.6023	0.2354	0.8141
θ_3	0.24788	0.4594	0.5394	0.5902
θ_4	0.0117	0.4766	0.0246	0.9804
θ_5	0.1086	0.3338	0.3253	0.7453
θ_6	0.0165	0.3131	0.0528	0.9579
θ_7	-0.2415	0.2276	-1.0612	0.2900
AIC	1.5530	SBIC	1.7233	
	C. ARMA(2,(1,7))			
c	1.3742	0.2259	6.0830	0.0000
ϕ_1	0.3123	0.0865	3.6072	0.0004
ϕ_2	0.6087	0.0860	7.0750	0.0000
θ_1	0.9125	0.0464	19.6576	0.0000
θ_7	-0.1676	0.0304	-5.5013	0.0000
AIC	1.5556	SBIC	1.6237	

Notes:

(1) In the case of the ARMA(2,6) specification, the roots of the characteristic equation associated to Φ are .54 and -.63. The roots of the characteristic equation associated to Θ are .48±.61i, -.24±.75i, and -.89±.24i.

(2) In the case of the ARMA(2,7) specification, the roots of the characteristic equation associated to Φ are .77 and -.16. The roots of the characteristic equation associated to Θ are .67, .45±.67i, -.24±.81i, and -.85±.25i.

(3) In the case of the ARMA(2,(1,7)) specification, the roots of the characteristic equation associated to Φ are .95 and -.64. The roots of the characteristic equation associated to Θ are .69, .39±.58i, -.28±.70i, and -.91±.24i.

Table 6: Estimation of selected models

Parameter	Estimation	Std. Err.	t-stat.	p-value
		A. AR(2)		
c	1.3876	0.2852	4.8649	0.0000
ϕ_1	1.1091	0.07083	15.6595	0.0000
ϕ_2	-0.2454	0.0708	-3.4663	0.0007
		B. AR(7)		
c	1.3892	0.2539	5.4706	0.0000
ϕ_1	1.1793	0.0743	15.8793	0.0000
ϕ_2	-0.4706	0.1118	-4.2072	0.0000
ϕ_3	0.3918	0.1147	3.4170	0.0008
ϕ_4	-0.3449	0.1156	-2.9838	0.0032
ϕ_5	0.3243	0.1148	2.8249	0.0053
ϕ_6	-0.3832	0.1118	-3.4265	0.0008
ϕ_7	0.1524	0.0744	2.0477	0.0421
		C. ARMA(1,1)		
c	1.4065	0.2824	4.9800	0.0000
ϕ_1	0.8127	0.04628	17.5634	0.0000
θ_1	0.3784	0.0739	5.1223	0.0000
		D. ARMA(2,1)		
c	1.3949	0.3178	4.3886	0.0000
ϕ_1	0.4411	0.1481	2.9777	0.0033
ϕ_2	0.3516	0.1414	2.4866	0.0138
θ_1	0.73019	0.1132	6.4486	0.0000
		E. ARMA(2,(1,7))		
ϕ_1	0.3123	0.0865	3.6072	0.0004
ϕ_2	0.6087	0.0860	7.0750	0.0000
θ_1	0.9125	0.0464	19.6576	0.0000
θ_2	-0.1676	0.0304	-5.5014	0.0000

Notes:

(1) In the case of the AR(2) specification, the roots of the characteristic equation associated to Φ are .80 and .31.

(2) In the case of the AR(7) specification, the roots of the characteristic equation associated to Φ are 0.66, .75±.22i , .11±.76i, and -.60±.54i.

(3) In the case of the ARMA(1,1) specification, the root of the characteristic equation associated to Φ is 0.81. The root of the characteristic equation associated to Θ is -0.38.

(4) In the case of the ARMA(2,1) specification, the roots of the characteristic equation associated to Φ are .85 and -.41. The root of the characteristic equation associated to Θ is -0.73.

(5) In the case of the ARMA(2,(1,7)) specification, the roots of the characteristic equation associated to Φ are .95 and -.64. The roots of the characteristic equation associated to Θ are .69, .39±.58i, -.28±.70i, and -.91±.24i.

Table 7: Autocorrelation of residuals

Lag	Autocorrelation	Q-stat	p-value
	B. AR(2)		
1	0.0225	0.0975	0.7548
2	-0.0990	1.9981	0.3682
3	0.1588	6.9205	0.0745
4	-0.0328	7.1318	0.1291
5	0.1403	11.014	0.0511
6	-0.0930	12.727	0.0476
7	-0.1617	17.938	0.0123
8	0.1352	21.599	0.0057
9	0.0542	22.192	0.0083
10	-0.0392	22.503	0.0127
⋮			
20	0.0169	36.035	0.0152
	B. AR(7)		
1	0.0066	0.0085	0.9267
2	0.0245	0.1253	0.9392
3	-0.0072	0.1356	0.9873
4	-0.0199	0.2129	0.9947
5	-0.0421	0.5626	0.9896
6	0.0217	0.6557	0.9954
7	-0.0575	1.3137	0.9881
8	0.1221	4.3041	0.8287
9	0.0767	5.4902	0.7897
10	0.0113	5.5159	0.8542
⋮			
20	0.0666	18.496	0.2248
	B. ARMA(1,1)		
1	-0.0305	0.1795	0.6717
2	-0.0169	0.2347	0.8892
3	-0.1708	5.9238	0.1154
4	-0.0696	6.8749	0.1426
5	0.1488	11.243	0.0468
6	-0.0899	12.845	0.0456
7	-0.1501	17.338	0.0153
8	0.1331	20.889	0.0074
9	0.0117	20.917	0.0130
10	-0.0162	20.970	0.0213
⋮			
20	0.0126	34.231	0.0246

Table 7 (cont'd): Autocorrelation of residuals

Lag	Autocorrelation	Q-stat	p-value
	B. ARMA(2,1)		
1	0.0130	0.0325	0.6717
2	0.0536	0.5908	0.8892
3	0.0713	1.5812	0.1154
4	-0.0089	1.5969	0.1426
5	0.0793	2.8363	0.0468
6	-0.0797	4.0969	0.0456
7	-0.1631	9.4001	0.0153
8	0.1271	12.638	0.0074
9	-0.0032	12.640	0.0130
10	-0.0074	12.651	0.0213
⋮			
20	0.0384	27.971	0.1101
	B. ARMA(2,(7,1))		
1	0.0439	0.3727	0.5417
2	0.0708	1.3452	0.5104
3	0.0223	1.4421	0.6957
4	0.0267	1.5820	0.8120
5	0.0141	1.6210	0.8987
6	-0.0438	2.0009	0.9196
7	-0.0748	3.1166	0.8740
8	0.0284	3.2783	0.9157
9	0.0852	4.7403	0.8563
10	-0.0908	6.4108	0.7796
⋮			
20	0.0610	20.503	0.4269

Table 8: Diebold and Mariano test (using RMSE)

M_0	M_1	RMSE (M_0)	RMSE (M_1)	DM-stat.	p-value
AR(7)	ARMA(2,(1,7))	0.2100	0.1911	0.9146	0.3604
	AR(2)	0.2100	0.1816	1.4120	0.1581
	ARMA(1,1)	0.2100	0.1817	1.4620	0.1437
	ARMA(2,1)	0.2100	0.1934	0.9333	0.3507
ARMA(2,(1,7))	AR(2)	0.1911	0.1816	0.8186	0.4130
	ARMA(1,1)	0.1911	0.1817	1.0630	0.2877
	ARMA(2,1)	0.1911	0.1934	-0.1950	0.8452
AR(2)	ARMA(1,1)	0.1816	0.1817	-0.0446	0.9644
	ARMA(2,1)	0.1816	0.1934	-1.0670	0.2860
ARMA(1,1)	ARMA(2,1)	0.1817	0.1934	-1.2330	0.2174

Table 9: Diebold and Mariano test (using MAD)

M_0	M_1	MAE (M_0)	MAE (M_a)	DM-stat.	p-value
AR(7)	ARMA(2,(1,7))	0.3587	0.3587	10^{-6}	0.9998
	AR(2)	0.3587	0.3392	2.4210	0.0155
	ARMA(1,1)	0.3587	0.3363	2.0560	0.0397
	ARMA(2,1)	0.3587	-	-	-
ARMA(2,(1,7))	AR(2)	0.3587	0.3392	1.8920	0.0584
	ARMA(1,1)	0.3587	0.3363	1.7600	0.0784
	ARMA(2,1)	0.3587	0.3519	0.5033	0.6148
AR(2)	ARMA(1,1)	0.3392	0.3363	0.7268	0.4673
	ARMA(2,1)	0.3392	0.3519	-2.2770	0.0228
ARMA(1,1)	ARMA(2,1)	0.3363	0.3519	-3.2460	0.0012

Table 10: Link between observed and forecasted values

Parameter	Estimation	Std. Err.	t-stat.	p-value
	A. AR(2)			
b	1.0554	0.06222	16.9600	0.0000
a	-0.0695	0.1031	-0.6700	0.5030
	B. AR(7)			
b	0.9897	0.0641	15.4300	0.0000
a	0.0343	0.1073	0.3200	0.7500
	C. ARMA(2,(1,7))			
b	1.0241	0.0627	16.330	0.0000
a	-0.0293	0.1048	-0.2800	0.7810
	D. ARMA(1,1)			
b	1.0571	0.0623	16.9600	0.0000
a	-0.0734	0.1032	-0.7100	0.4800
	D. ARMA(1,1)			
b	1.0344	0.0636	16.2400	0.0000
a	-0.0421	0.1060	-0.4000	0.6930

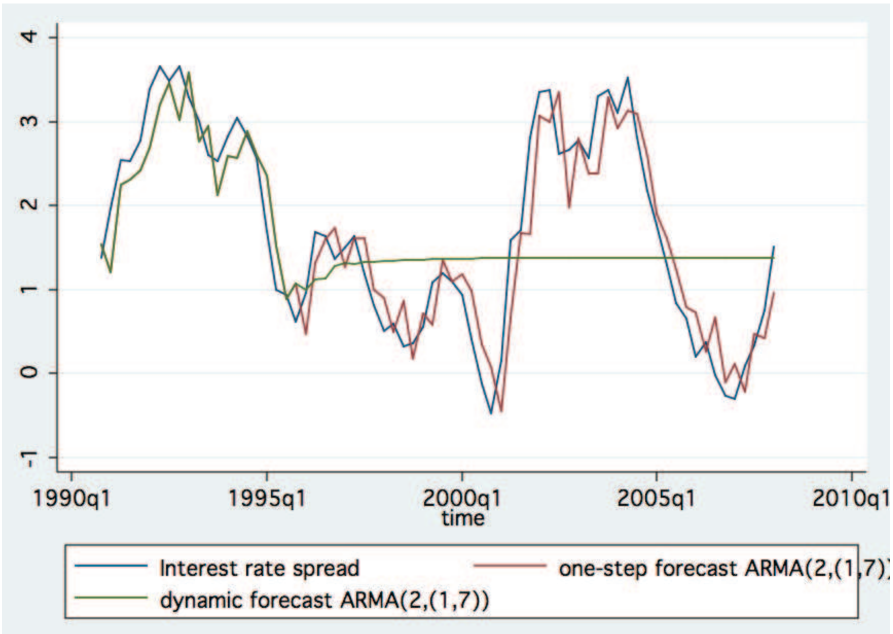

Figure 6: Forecasts

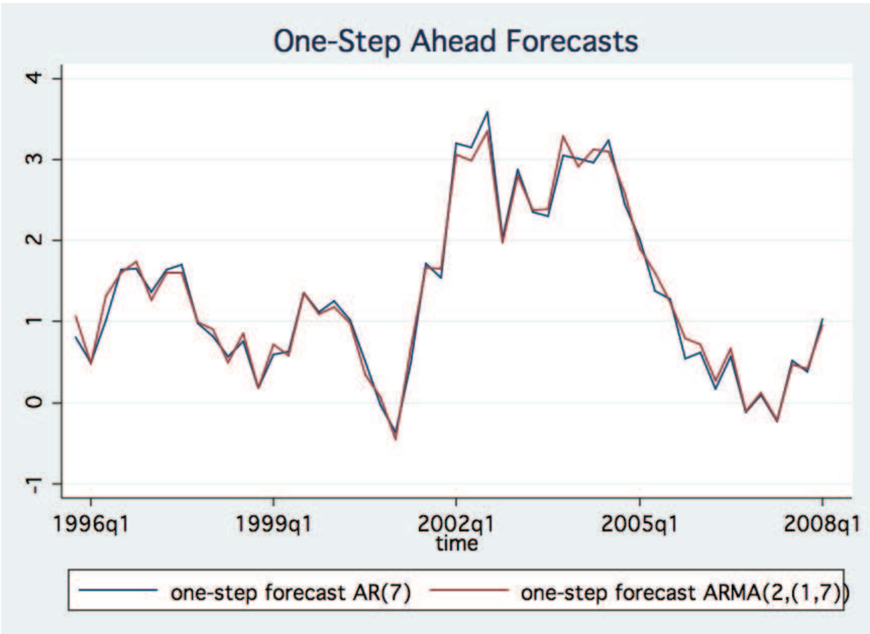

Figure 7: Forecasts