Master in Financial Engineering (EPFL)

Financial Econometrics

Some elements of correction: GARCH models

Exercise 1: Let $(X_t) \sim \texttt{GARCH}(p, q)$ such that

$$\begin{aligned} X_t &= \sigma_t Z_t \\ Z_t &\sim \text{ i.i.d. } \mathcal{N}(0,1) \\ \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \end{aligned}$$

where the usual conditions on the parameters hold true.

1. The result is straightforward since σ_t^2 does depend on past values that are known given the information set I_{t-1} and $Z_t^2 \sim \chi^2(1)$ (with unconditional expectation 1).

2. One has

$$\sigma_t^2 = X_t^2 - \eta_t$$

and

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

Therefore

$$X_t^2 - \eta_t = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \left(X_{t-j}^2 - \eta_{t-j} \right).$$

Consequently, if $p > q$, then:

$$X_t^2 = \omega + \sum_{i=1}^{q} \left(\alpha_i + \beta_i\right) X_{t-i}^2 + \sum_{j=q+1}^{p} \alpha_i X_{t-i}^2 + \eta_t - \sum_{i=1}^{q} \beta_i \eta_{t-i}.$$

Otherwise (if $q > p$):

$$X_t^2 = \omega + \sum_{i=1}^{p} \left(\alpha_i + \beta_i\right) X_{t-i}^2 + \sum_{j=p+1}^{q} \beta_i X_{t-i}^2 + \eta_t - \sum_{i=1}^{q} \beta_i \eta_{t-i}.$$

Finally,

$$X_t^2 = \omega + \sum_{i=1}^{r} (\alpha_i + \beta_i) X_{t-1}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}.$$

where $r = \max\{p, q\}$, $\alpha_i = 0$ if $i > p$, and $\beta_j = 0$ if $j > q$.

3. It follows that $(X_t^2) \sim \texttt{ARMA}(r,q)$ [...]

Exercise 2: Let $(X_t) \sim \texttt{GARCH}(1,1)$ such that

$$\begin{aligned} X_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

where $Z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$, $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$ (with $\alpha + \beta < 1$). We further assume that $1 - 3\alpha^2 - \beta^2 - 2\alpha\beta > 0$.

1. (X_t^2) has the following $\texttt{ARMA}(1,1)$ representation (see Exercise 1):

$$X_t^2 - (\alpha+\beta) X_{t-1}^2 = \omega + v_t - \beta v_{t-1}$$

where $v_t = X_t^2 - \mathbb{E}\left[X_t^2 \mid X_u, u < t\right]$ is a (weak) white noise. Consequently, using the results of Lecture 4 part 2.b and exercise session 5, the autocorrelation function of (X_t^2) satisfies, for all $h > 1$:

$$\rho_{X^2}(h) = (\alpha+\beta)\rho_{X^2}(h-1).$$

It remains determining the first two autocovariances. This can be done by using the infinite order moving average representation. Indeed, one has

$$X_t^2 = \frac{\omega}{1-\alpha-\beta} + v_t + \alpha \sum_{i=1}^{\infty} (\alpha+\beta)^{i-1} v_{t-i}.$$

and

$$\begin{aligned} \gamma_{X^2}(0) &= \left(1 + \alpha^2 \sum_{i=1}^{\infty} (\alpha+\beta)^{2(i-1)}\right) \mathbb{E}\left[v_t^2\right] = \left(1 + \frac{\alpha^2}{1-(\alpha+\beta)^2}\right) \mathbb{E}\left[v_t^2\right] \\ \gamma_{X^2}(1) &= \left(\alpha + \alpha^2(\alpha+\beta) \sum_{i=1}^{\infty} (\alpha+\beta)^{2(i-1)}\right) \mathbb{E}\left[v_t^2\right] \\ &= \left(\alpha + \frac{\alpha^2(\alpha+\beta)}{1-(\alpha+\beta)^2}\right) \mathbb{E}\left[v_t^2\right]. \end{aligned}$$

where

$$\mathbb{E}\left[v_t^2\right] = \mathbb{E}\left[\left(X_t^2 - \sigma_t^2\right)^2\right] = \mathbb{E}\left[\left(Z_t^2 - 1\right)^2\right] \times \mathbb{E}\left[\sigma_t^4\right] = 2\mathbb{E}\left[\sigma_t^4\right]$$

with $\mathbb{E}\left[\sigma_t^4\right]$ satisfying the following relationship

$$\begin{aligned} \mathbb{E}\left[\sigma_t^4\right] &= \mathbb{E}\left[(\omega + \alpha X_{t-1}^2 + \beta \sigma_t^2)^2\right] \\ &= \omega^2 + 3\alpha^2 \mathbb{E}\left[\sigma_t^4\right] + \beta^2 \mathbb{E}\left[\sigma_t^4\right] + 2\omega(\alpha+\beta)\mathbb{E}\left[\sigma_t^2\right] + 2\alpha\beta \mathbb{E}\left[\sigma_t^4\right] \end{aligned}$$

and thus

$$\mathbb{E}\left[\sigma_t^4\right] = \frac{\omega^2(1+\alpha+\beta)}{(1-\alpha-\beta)(1-3\alpha^2-\beta^2-2\alpha\beta)}.$$

Finally, the first autocorrelation is then $\rho_{X^2}(1) = \frac{\alpha(1-\beta^2-\alpha\beta)}{1-\beta^2-2\alpha\beta}$.

Exercise 3:

1. One could test (say, at 5% significance level)

$$H_0 : \alpha_1 = 0$$
$$H_a : \alpha_1 \neq 0$$

Note that the alternative hypothesis can be written as $H_a : \alpha_1 > 0$ (taking into account the parametric restriction of an ARCH(1) model). One can use any ML-based test (e.g., Wald test).

2. One key issue is to determine an estimate of the asymptotic variance-covariance matrix of the ML estimator (of α_1). This can be achieved as follows. First, note that the log-likelihood function is given by:

$$\ell(\theta \mid \{r_t, t = 2, \cdots, T\} = -\frac{1}{2}\sum_{t=2}^{T}\left\{\ln(2\pi) + \ln(\sigma_t^2) - \frac{r_t^2}{\sigma_t^2}\right\}$$

where

$$\sigma_t^2 = 0.0001 + \alpha_1 r_{t-1}^2.$$

It follows that

- The first-order derivative is given by:

$$\frac{d\ell}{d\alpha_1}(.) = -\frac{1}{2}\left\{\sum_{t=2}^{T}\frac{r_{t-1}^2}{\sigma_t^2} - \sum_{t=2}^{T}\frac{r_t^2}{\sigma_t^4}r_{t-1}^2\right\}$$

- The second-order derivative is given by:

$$\frac{d^2\ell}{d\alpha_1^2}(.) = \sum_{t=2}^{T}\frac{r_{t-1}^4}{2\sigma_t^4} - \sum_{t=2}^{T}\frac{r_t^2 r_{t-1}^4}{\sigma_t^6}.$$

Using an estimate of the second-order derivative, one gets:

$$\frac{d^2\ell}{d\alpha_1^2}(.)|_{\alpha_1=\widehat{\alpha}_1} = -1606.032$$

It follows that an estimate of the asymptotic variance-covariance matrix is given by:

$$-\left[\frac{d^2\ell}{d\alpha_1^2}(.)|_{\alpha_1=\widehat{\alpha}_1}\right]^{-1} = -\frac{1}{-1606.032} = 0.00062$$

Accordingly,

$$\widehat{\alpha}_1 \overset{a}{\sim} \mathcal{N}(0, 0.00062).$$

or alternatively

$$\frac{\widehat{\alpha}_1}{\sqrt{0.00062}} \overset{a}{\sim} \mathcal{N}(0, 1).$$

It is straightforward to show that one can reject the null hypothesis at 5%.

3. Using Appendix 1, one has

$$\texttt{VaR}_{90\%}^{T+1} = -\sqrt{0.0001 + 0.3577 \times (-0.00066)^2} \times (-1.282) = 0.0128$$

$$\texttt{VaR}_{95\%}^{T+1} = -\sqrt{0.0001 + 0.3577 \times (-0.00066)^2} \times (-1.645) = 0.0165$$

$$\texttt{VaR}_{99\%}^{T+1} = -\sqrt{0.0001 + 0.3577 \times (-0.00066)^2} \times (-2.326) = 0.0233$$

Exercise 4:

1. Using Appendix 1, the GARCH(1,1)-based VaR forecast at date $T+1$ is (minus) the α^{th} quantile of the conditional distribution of $r_{T+1} \mid \underline{r_T}$. In a GARCH(1,1) specification, one has

$$r_{T+1} \mid \underline{r_T} \sim \mathcal{N}\left(0, \omega + \alpha_1 r_T^2 + \beta_1 \sigma_T^2\right).$$

Accordingly,

$$\begin{aligned} \texttt{VaR}_{1-\alpha}^{T+1} &= -\alpha\texttt{th}\text{quantile of } \mathcal{N}\left(0, \omega + \alpha_1 r_T^2 + \beta_1 \sigma_T^2\right) \\ &= -\sqrt{\omega + \alpha_1 r_T^2 + \beta_1 \sigma_T^2} \times z_\alpha \end{aligned}$$

where z_α is the quantile of order α of standard normal random variable. Using the estimates, one gets

$$\texttt{VaR}_{90\%}^{T+1} = 0.0078$$

$$\texttt{VaR}_{95\%}^{T+1} = 0.0100$$

$$\texttt{VaR}_{99\%}^{T+1} = 0.0141$$

2. Using the historical simulations method, one determines the quantile of order 0.1 using the data set of 20 observations. In so doing, one needs to find the position of this quantile (in the order sample) : $(n+1) \times \alpha$ where $n = 20$ is the sample siez and α the quantile order. The corresponding position is 2.1
Using a standard rule of thumb (i.e., the simple average of the adjacent values (corresponding to the position 2 and 3), one gets:

$$\texttt{VaR}_{90\%}^{\texttt{HS},T+1} = -\frac{-0.00351 - 0.00317}{2} = 0.00334$$

Exercise 5: Let $(\epsilon_t) \sim \texttt{GARCH}(1,1)$ such that

$$\begin{aligned} \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

where $Z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$, $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$ (with $\alpha + \beta < 1$).

1. By definition, the one-step ahead forecast of σ_t^2 is given by:

$$\sigma_t^2(1) = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2.$$

Using the expression of the conditional variance at time $t+2$

$$\sigma_{t+2}^2 \;=\; \omega + (\alpha+\beta)\,\sigma_{t+1}^2 + \alpha\sigma_{t+1}^2(Z_{t+1}^2 - 1),$$

the two-step ahead forecast of the conditional variance is given by:

$$\begin{aligned}\mathbb{E}\left[\sigma_{t+2}^2 \mid I_t\right] &= \omega + (\alpha+\beta)\,\mathbb{E}\left[\sigma_{t+1}^2 \mid I_t\right] \\ &\quad +\alpha\mathbb{E}\left[\sigma_{t+1}^2(Z_{t+1}^2 - 1) \mid I_t\right]\end{aligned}$$

i.e.

$$\sigma_t^2(2) = \omega + (\alpha+\beta)\,\sigma_t^2(1)$$

since $\mathbb{E}\left[\sigma_{t+1}^2(Z_{t+1}^2 - 1) \mid I_t\right] = 0$. More generally, using

$$\sigma_{t+h}^2 \;=\; \omega + (\alpha+\beta)\,\sigma_{t+h-1}^2 + \alpha\sigma_{t+h-1}^2(Z_{t+h-1}^2 - 1),$$

the h-step ahead forecast of the conditional variance is given by:

$$\begin{aligned}\mathbb{E}\left[\sigma_{t+h}^2 \mid I_t\right] &= \omega + (\alpha+\beta)\,\mathbb{E}\left[\sigma_{t+h-1}^2 \mid I_t\right] \\ &\quad +\alpha\mathbb{E}\left[\sigma_{t+h-1}^2(Z_{t+h-1}^2 - 1) \mid I_t\right]\end{aligned}$$

i.e.

$$\sigma_t^2(h) = \omega + (\alpha+\beta)\,\sigma_t^2(h-1)$$

since $\mathbb{E}\left[\sigma_{t+h-1}^2(Z_{t+h-1}^2 - 1) \mid I_t\right] = 0$. Then, by backward substitution, one obtains

$$\sigma_t^2(h) = \frac{\omega\left(1-(\alpha+\beta)^{h-1}\right)}{1-(\alpha+\beta)} + (\alpha+\beta)^{h-1}\,\sigma_t^2(1)$$

with

$$\sigma_{t+1}^2 \equiv \sigma_t^2(1) = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2.$$

2. As $h \to \infty$, one has:

$$\sigma_t^2(h) \underset{h\to\infty}{\to} \frac{\omega}{1-(\alpha+\beta)}$$

This is the unconditional variance of ϵ_t^2.

3. Suppose now that one considers an AR(1) process with GARCH(1,1) error terms:

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{aligned}$$

where $|\phi| < 1$.

3.1. Find $\mathbb{V}[X_{t+h} \mid X_u, u < t]$ if $\phi^2 \neq \alpha+\beta$ (respectively, $\phi^2 = \alpha+\beta$)).
One has

$$\begin{aligned}\mathbb{V}[X_{t+h} \mid X_u, u < t] &= \mathbb{V}\left[\sum_{i=0}^{h} \phi^{h-i}\epsilon_{t+i} \mid \epsilon_u, u < t\right] \\ &= \sum_{i=0}^{h} \phi^{2(h-i)} \mathbb{V}[\epsilon_{t+i} \mid \epsilon_u, u < t].\end{aligned}$$

Using the result of Question 1, if $\phi^2 \neq \alpha+\beta$, one has:

$$\begin{aligned}\mathbb{V}[X_{t+h} \mid X_u, u < t] &= \frac{\omega}{1-(\alpha+\beta)}\left(\sum_{i=0}^{h}\phi^{2(h-i)}\right) \\ &\quad + \left(\sigma_t^2 - \frac{\omega}{1-(\alpha+\beta)}\right)\sum_{i=0}^{h}(\alpha+\beta)^i\phi^{2(h-i)} \\ &= \frac{\omega\left(1-\phi^{2(h+1)}\right)}{(1-(\alpha+\beta))(1-\phi^2)} \\ &\quad + \left(\sigma_t^2 - \frac{\omega}{1-(\alpha+\beta)}\right)\frac{\phi^{2(h+1)}-(\alpha+\beta)^{h+1}}{\phi^2-(\alpha+\beta)}.\end{aligned}$$

If $\phi^2 = \alpha+\beta$, then

$$\mathbb{V}[X_{t+h} \mid X_u, u < t] = \frac{\omega\left(1-\phi^{2(h+1)}\right)}{(1-\phi^2)^2} + \left(\sigma_t^2 - \frac{\omega}{1-(\alpha+\beta)}\right)(h+1)\phi^{2h}.$$

3.2. The term $\sigma_t^2 - \frac{\omega}{1-(\alpha+\beta)}$ captures the difference between the conditional and the unconditional variance of the error terms. The coefficient of this term always being positive, the variance of the prediction at horizon h increases linearly with respect to the difference between the conditional and unconditional variance. In this respect, a large negative difference (e.g., in the case of a low-volatility period) will lead to more accurate predictions. In contrast, it will tend to deteriorate when σ_t^2 is large. Finally, as $h \to \infty$, the contribution of this factor decreases and one retrieves the unconditional variance of X_t:

$$\lim_{h\to\infty} \mathbb{V}[X_{t+h} \mid X_u, u < t] = \frac{\mathbb{V}(\epsilon_t)}{1-\phi^2}.$$

3.3. If $\mid \phi \mid = 1$ and the process is initialized at 0, one has

$$\mathbb{V}[X_{t+h} \mid X_u, u < t] = \frac{\omega(1+h)}{(1-(\alpha+\beta))} + \left(\sigma_t^2 - \frac{\omega}{1-(\alpha+\beta)}\right)\frac{1-(\alpha+\beta)^{h+1}}{1-(\alpha+\beta)}.$$

In contrast to the previous question, the impact of the observations (before time t) does not vanish as h increases (second right-hand side term). However, this term is dominated (and thus becomes negligible) by the deterministic part (first right-hand side term) which is proportional to h.

Exercise 6:

1. Comment the different specifications: See Lecture Notes
2. Some comments:
   - All models are (weakly) stationary.
   - `GARCH`: (usual interpretation of α and β). The variance and kurtosis in the estimated model are respectively given by 1.3×10^{-4} and 3.49. Given the `ARMA`$(1,1)$ representation, one has (see Exercise 2!)

   $$\rho_{\epsilon^2}(h) = (\alpha + \beta)\rho_{\epsilon^2}(h-1).$$

   Since $\hat{\alpha} + \hat{\beta} \simeq 0.93$, there is a slow decay of the autocorrelation function.
   - `EGARCH`: Since the estimate of θ is negative, there is the presence of the leverage effect. This is also true for the `QGARCH` (negative sign of the coefficient estimate of ϵ_{t-1}) and the `GJR-GARCH` model (negative sign of the coefficient estimate of $\Pi^{-}_{t-1}\epsilon^2_{t-1}$).
   - The `TGARCH` model also displays the leverage effect. The term $\omega/(1-\beta)$ can be interpreted as the "minimal volatility" after assuming that all the innovations are equal to zero. The estimates of α_1 and α_2 represent the impact of the last observation (after controlling the sign) on the current volatility. Notably, the effect of a negative value is 3.5 more than a positive value. The last coefficient, which captures the importance of the last volatility, [...]
3. Some comments
   - Note that the log-likelihood value of the `GARCH`$(1,1)$ cannot be compared directly with that of other models since it has one parameter less!
   - The asymmetric `GARCH` models can be compared (they all have five parameters).
   - The largest value is observed for the `GJR-GARCH` model.
   - However, the difference (in terms of ℓ) between these asymmetric models is so small, that it is not so clear which model is really better (superior) than the others;
   - Log-likelihood ratio tests [...]