Master in Financial Engineering, EPFL

Course: Financial Econometrics

Case study 2: GARCH models

Maxime Couvert & Florian Pelgrin

Exercise 1

The main objective of this exercise is to get a suitable model for a given log-return using daily information. In so doing, one considers the following four specifications.

- Model 1: Constant model

  $$r_t = \mu + \epsilon_t$$

  where ϵ_t is a weak white noise with expectation zero and with variance σ_ϵ^2.[1]

- Model 2: AutoRegressive process of order 1—AR(1)

  $$r_t = \mu + \phi r_{t-1} + \epsilon_t$$

  where $\mid \phi \mid < 1$ and ϵ_t is a weak white noise with expectation zero and variance σ_ϵ^2.

- Model 3: AutoRegressive and Conditionnally Heteroskedastic process of order p—ARCH(p):

  $$\begin{aligned} r_t &= \mu + \epsilon_t \\ \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 \end{aligned}$$

  where $Z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$.

- Model 4: Generalized AutoRegressive and Conditionnally Heteroskedastic process of orders p and q—GARCH(p,q):

  $$\begin{aligned} r_t &= \mu + \epsilon_t \\ \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \end{aligned}$$

  where $Z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$.

[1] The stochastic process $(\epsilon_t)_t$ is said to be a weak white noise if (i) $\mathbb{E}[\epsilon_t] = 0$ for all t, (ii) $\mathbb{V}[\epsilon_t] = \sigma_\epsilon^2 < +\infty$ for all t, and (iii) $\mathbb{C}ov[\epsilon_t, \epsilon_{t'}] = 0$ when $t \neq t'$.

Using the excel file, `data_case_study_2_epfl_2019.xls`, pick one stock at random and answer the following questions.

- **Part I**:
  1.1. Provide an OLS estimation of Model 1.
  1.2. Plot the residuals. What do you observe? Is it consistent with stylized facts of Lecture 1?
- **Part II**:
  2.1. Plot the autocorrelation function and the partial autocorrelation function of r_t for $k = 1, \cdots, 20$.
  2.2. Using the results of the previous question, does it make sense to consider an autoregressive process of order 1. Explain carefully.
  2.3. Provide an OLS estimation of Model 1 (OLS estimates, standard errors, p-values) and interpret the results.
- **Part III**:
  3.1. Plot the autocorrelation function of the squared (demeaned)log-return for $k = 1, \cdots, 20$. Interpret the results. Notably, What does the squared (demeaned) log-return capture? Is it consistent with some of the stylized facts of Lecture 1?
  3.2. Interpret the different equations of Model 3.
  3.3. Estimate the ARCH(1) specification.
    * Interpret the results.
    * Plot the residuals ($\hat{\epsilon}_t$) and the standardized (or filtered) residuals ($\hat{\epsilon}_t/\hat{\sigma}_t$) and their corresponding empirical distributions. Interpret the different results.

  3.4. Proceed as in Question 3.3. with $p = 4$ and 10. Interpret the results. Notably, what happens when the number of lags increases?
- **Part IV**:
  4.1. Interpret the different equations of Model 4.
  4.2. Compute the EWMA-based variance of the log-return using a fixed window of 90 days:

  $$\sigma_t^2 = 0.94 \times \sigma_{t-1}^2 + 0.06 r_t^2$$

  where σ_0^2 is the (empirical) variance of the log-return for the first 90 observations.
  4.3. Provide an ML-based estimation of a GARCH(1,1) specification.
    * Interpret the estimation results. What is the unconditional volatility?

   * How does it compare with the results of Question 4.2.?
   * Plot the residuals ($\hat{\epsilon}_t$) and the standardized (or filtered) residuals ($\hat{\epsilon}_t/\hat{\sigma}_t$) and their corresponding empirical distributions. Interpret the different results.

4.4. Proceed as in Question 4.3. with $p = 3$ and $q = 3$. Interpret the results. Notably, what happens when the number of lags for p and q increases?

- How can one use all of the information of this exercise?

Exercise 2: Term structure of volatility

The main goal of this exercise is to determine the volatility term structure using the information provided in the Excel file Term_structure_of_volatility_epfl_2019.xls
In so doing, consider the simple return model with GARCH(1,1) error terms

$$\begin{aligned} r_t &= \mu + \epsilon_t \\ \epsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1} \end{aligned}$$

where $z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$, $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\alpha + \beta_1 < 1$. It is worth noting that :

$$\mathbb{E}\left[\epsilon_{t+h} \mid \epsilon_\tau, \tau < t\right] = \mathbb{E}\left[\mathbb{E}\left[\epsilon_{t+h} \mid \underline{\epsilon_{t+h-1}}\right] \mid \epsilon_\tau, \tau < t\right] = 0$$

and

$$\mathbb{E}\left[\epsilon_{t+h}^2 \mid \epsilon_\tau, \tau < t\right] = \mathbb{E}\left[\sigma_{t+h}^2 \mid \epsilon_\tau, \tau < t\right]$$

or equivalently

$$\mathbb{E}\left[\epsilon_{t+h}^2 \mid I_\tau, \tau < t\right] = \mathbb{E}\left[\sigma_{t+h}^2 \mid I_\tau, \tau < t\right].$$

In the sequel, the h-step ahead forecast of σ_t^2 given the information available at period t is denoted:

$$\mathbb{E}\left[\sigma_{t+h}^2 \mid I_{t-1}\right] \equiv \mathbb{E}_{t-1}\left[\sigma_{t+h}^2\right] \equiv \sigma_{t+h|t-1}^2.$$

Determination of forward variance forecasts

The term structure of (daily) volatility is determined by using the (daily) forward variance forecasts

$$\left\{\sigma_{t|t-1}^2, \sigma_{t+1|t-1}^2, \cdots, \sigma_{t+h|t-1}^2\right\}.$$

Show that

- The h-step ahead forecast of the conditional variance is defined by the relationship (for all $h \geq 1$):

$$\mathbb{E}_{t-1}\left[\sigma_{t+h}^2\right] = \omega + (\alpha_1 + \beta_1)\,\mathbb{E}_{t-1}\left[\sigma_{t+h-1}^2\right]$$

  or, equivalently,

$$\mathbb{E}_{t-1}\left[\sigma_{t+h}^2\right] = \omega \frac{1 - (\alpha_1 + \beta_1)^h}{1 - (\alpha_1 + \beta_1)} + (\alpha_1 + \beta_1)^h \sigma_t^2.$$

- The best conditional forecast as $h \to \infty$ is given by

$$\mathbb{E}_{t-1}\left[\sigma_{t+h}^2\right] \underset{h \to \infty}{\to} \bar{\sigma}^2 = \frac{\omega}{1 - \alpha_1 - \beta_1}.$$

Determination of the cumulative forward variance

Taking the (daily) forward variance forecasts, determine the average volatility over the next h days (assuming that there are 252 trading days).

Application

Using the Excel file, and especially the financial series FTSE100 over the period Dec. 31, 2003- August 29, 2007:

- Provide some descriptive statistics of the corresponding (log-) return;
- Compute the EWMA volatility;
- Estimate a GARCH(1,1) model;
- Compute the forward variance (forecast of conditional variances for different horizons);
- Calculate the cumulative forward variance and plot the corresponding term structure of volatility;
- Compare the previous results with those of a constrained GARCH(1,1) model (i.e., after imposing a long-term volatility constraint).