Exercise Three

The following definitions were discussed in Buro's lecture last week.

Let $\omega_1 = \sum_I \alpha_I d\omega^I$ be a differential k-form on $\mathbb{R}^n$ and $\omega_2 = \sum_J \beta_J dx^J$ a differential s-form on $\mathbb{R}^n$, where $I \in \mathcal{I}_k$, $J \in \mathcal{I}_s$ are ordered k-tuple and ordered s-tuple. Then the *wedge product* of ω_1 and ω_2 is defined to be a differential $(k+s)$-form by

$$\omega_1 \wedge \omega_2 = \sum_{I,J} \alpha_I \beta_J dx^I \wedge dx^J.$$

Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Let ω be a differential k-form in $\mathbb{R}^m$. Then f induces a differential k-form $f^*\omega$ in $\mathbb{R}^n$, called the *pull back of ω under f*, as follows

$$f^*\omega(p)(v_1, \cdots, v_k) = \omega(f(p))(df_p(v_1), \cdots, df_p(v_k)) \qquad \text{if } k \geq 1$$

and

$$f^*(\omega) = \omega \circ f \qquad \text{if } k = 0.$$

Above, $df_p \colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is the differential of f at $p \in \mathbb{R}^n$.

1. Show that the definition of wedge product as above implies that if $\varphi_1, \cdots, \varphi_k$ are 1-forms on $\mathbb{R}^n$, then the wedge product of $\varphi_1 \wedge \cdots \varphi_k$ defines a differential k-form on $\mathbb{R}^n$ satisfying

$$\varphi_1 \wedge \cdots \wedge \varphi_k(v_1, \cdots, v_k) = \det\big(\varphi_i(v_j)\big) \quad i, j = 1, \cdots, k.$$

2. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Let ω_1 and ω_2 be differential k-forms in $\mathbb{R}^m$ and $g \colon \mathbb{R}^m \to \mathbb{R}$ a 0-form. Show that

- $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
- $f^*(g\omega_1) = f^*(g) f^*(\omega_1)$
- If $\theta_1, \cdots, \theta_k$ are 1-forms on $\mathbb{R}^m$, then $f^*(\theta_1 \wedge \cdots \wedge \theta_k) = f^*\theta_1 \wedge \cdots \wedge f^*\theta_k$.

3. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and $g \colon \mathbb{R}^l \to \mathbb{R}^n$ a smooth map. Show that

- $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$
- $(f \circ g)^*\omega = g^*\big(f^*(\omega)\big)$.

4. Show that the exterior derivative has the following basic properties:

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ whenever ω_1, ω_2 are two forms on $\mathbb{R}^n$
- If ω is a k-form on $\mathbb{R}^n$, then $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$
- If $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and ω is a k-form on $\mathbb{R}^m$, then $d(f^*\omega) = f^*(d\omega)$.

5. Consider the *winding form*

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

defined on $U = \mathbb{R}^2\backslash\{0\}$. Show that ω is a closed form on U. Recall that we know from Exercise Two 4 that it is not exact.