

# AN INTRODUCTION TO RIEMANNIAN GEOMETRY

CHANGYU GUO

## CONTENTS

1. Introduction	1
2. Calculus in Euclidean spaces	3
2.1. Functions and Taylor expansions	3
2.2. Tensor fields	5
2.3. Vector fields and differential forms	6
2.4. Riemannian metrics	13
2.5. Geodesics	15
2.6. Integration and inner products	19
2.7. Laplace-Beltrami operator	24
References	26

## 1. INTRODUCTION

**Question 1.1.** What kinds of quantities and operations appear in relation to analysis (or multivariable calculus) in a bounded open set  $U \subset \mathbb{R}^n$ ?

Some possible answers:

- Functions: continuity, partial derivatives, integrals,  $L^p$  spaces, Taylor expansions, Fourier or related expansions
- Vector fields: gradient, curl, divergence
- Measures, distributions, flows
- Laplace operator, Laplace, heat and wave equations
- Integration by parts formulas (Gauss, divergence, Green)
- Tensor fields, differential forms
- Distance, distance-minimizing curves (line segments), area, volume, perimeter

Imagine similar concepts on a hypersurface (e.g. double torus in  $\mathbb{R}^3$ )

This course is an introduction to analysis on manifolds. The first part of the course title has the following Wikipedia description: “[Mathematical Analysis](#) is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from

---

*Date:* March 20, 2019.

geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space).”

Following this description, our purpose will be to study in particular differentiation, integration, and differential equations on spaces that are more general than the standard Euclidean space  $\mathbb{R}^n$ . Different classes of spaces allow for different kinds of analysis:

- *Topological spaces* are a good setting for studying continuous functions and limits, but in general they do not have enough structure to allow studying derivatives
- The smaller class of *metric spaces* admits certain notions of differentiability, but in particular higher order derivatives are not always well defined
- *Differentiable manifolds* are modeled after pieces of Euclidean space and allow differentiation and integration, but they do not have a canonical Laplace operator and thus the theory of differential equations is limited

The class of spaces studied in this course will be that of Riemannian manifolds. These are differentiable manifolds with an extra bit of structure, a Riemannian metric, that allows to measure lengths and angles of tangent vectors. Adding this extra structure leads to a very rich theory where many different parts of mathematics come together. We mention a few related aspects, and some of these will be covered during this course (the more advanced topics that will be covered will be chosen according to the interests of the audience):

- (1) *Calculus*. Riemannian manifolds are differentiable manifolds, hence the usual notions of multivariable calculus on differentiable manifolds apply (derivatives, vector and tensor fields, integration of differential forms)
- (2) *Metric geometry*. Riemannian manifolds are metric spaces: there is a natural distance function on any Riemannian manifold such that the corresponding metric space topology coincides with the usual topology. Distances are realized by certain distinguished curves called geodesics, and these can be studied via a second order ODE (the geodesic equation).
- (3) *Measure theory*. Any oriented Riemannian manifold has a canonical measure given by the volume form. The presence of this measure allows to integrate functions and to define  $L^p$  spaces on Riemannian manifolds.
- (4) *Differential equations*. There is a canonical Laplace operator on any Riemannian manifold, and all the classical linear partial differential equations (Laplace, heat, wave) have natural counterparts
- (5) *Dynamical systems*. The geodesic flow on a closed Riemannian manifold is a Hamiltonian flow on the cotangent bundle, and the geometry of the manifold is reflected in properties of the flow (such as complete integrability or ergodicity)
- (6) *Conformal geometry*. The notions of conformal and quasiconformal mappings make sense on Riemannian manifolds, and there is enough underlying structure to provide many tools for studying them
- (7) *Topology*. There are several ways of describing topological properties of the underlying manifold in terms of analysis. In particular, Hodge theory characterizes

the cohomology of the space via the Laplace operator acting on differential forms, and Morse theory describes the topological type of the space via critical points of a smooth function on it

- (8) *Curvature.* The notion of curvature is fundamental in mathematics, and Riemannian manifolds are perhaps the most natural setting for studying curvature. Related concepts include the Riemann tensor, the Ricci tensor, and scalar curvature. There has been recent interest in lower bounds for Ricci curvature and their applications
- (9) *Inverse problems.* Many interesting inverse problems have natural formulations on Riemannian manifolds, such as integral geometry problems where one tries to determine a function from its integrals over geodesics, or spectral rigidity problems where one tries to determine properties of the underlying space from knowledge of eigenvalues of the Laplacian.
- (10) *Geometric analysis.* There are many branches of mathematics that are called geometric analysis. One particular topic is that of geometric evolution equations, where geometric quantities evolve according to a certain PDE. One of the most famous such equations is Ricci flow, where a Riemannian metric is deformed via its Ricci tensor. This was recently used by Perelman to complete Hamilton's program for proving the Poincaré and geometrization conjectures.

## 2. CALCULUS IN EUCLIDEAN SPACES

Let  $U$  be any nonempty open subset of  $\mathbb{R}^n$  (not necessarily bounded, and could be equal to  $\mathbb{R}^n$ ). We fix standard Cartesian coordinates  $x = (x_1, \dots, x_n)$  and will use these coordinates throughout this chapter. We may sometimes write  $x^j$  instead of  $x_j$ , and we will also denote by  $v_j$  or  $v^j$  the  $j$ -th coordinate of a vector  $v \in \mathbb{R}^n$ .

**2.1. Functions and Taylor expansions.** Let  $C(U)$  be the set of continuous functions on  $U$ . For partial derivatives, we will write

$$\partial_j f = \frac{\partial f}{\partial x_j} \quad \text{and} \quad \partial_{j_1 \dots j_k} f = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}.$$

We denote by  $C^k(U)$  the set of  $k$  times continuously differentiable real valued functions on  $U$ . Thus

$$C^k(U) = \left\{ f: U \rightarrow \mathbb{R} : \partial_{j_1 \dots j_l} f \in C(U) \text{ whenever } l \leq k \text{ and } j_1, \dots, j_l \in \{1, \dots, n\} \right\}.$$

Recall also that if  $f \in C^k(U)$ , then  $\partial_{j_1 \dots j_k} f = \partial_{j_{\sigma(1)} \dots j_{\sigma(k)}} f$  for any permutation  $\sigma$  of  $\{1, \dots, k\}$ .

We also denote by  $C^\infty(U)$  the infinitely differentiable functions on  $U$ , that is,

$$C^\infty(U) = \bigcap_{k \geq 0} C^k(U).$$

**Theorem 2.1** (Taylor expansion). Let  $f \in C^k(U)$ , let  $x_0 \in U$ , and assume that  $B(x_0, r) \subset U$ . If  $x \in B(x_0, r)$ , then

$$f(x) = \sum_{l=0}^k \frac{1}{l!} \left[ \sum_{j_1, \dots, j_l} \partial_{j_1 \dots j_l} f(x_0) (x - x_0)_{j_1} \cdots (x - x_0)_{j_l} \right] + R_k(x; x_0),$$

where  $|R_k(x; x_0)| \leq \eta(|x - x_0|)|x - x_0|^k$  for some function  $\eta$  with  $\eta(s) \rightarrow 0$  as  $s \rightarrow 0$ .

**Remark 2.2.** The Taylor expansion of order 2 is given by

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2} \nabla^2 f(x_0) (x - x_0) \cdots (x - x_0) + R_2(x; x_0),$$

where  $\nabla f = (\partial_1 f, \dots, \partial_n f)$  is the gradient of  $f$  and  $\nabla^2 f(x) = (\partial_{j_k} f(x))_{j, k=1}^n$  is the Hessian matrix of  $f$ .

*Proof.* Considering  $g(y) := f(x_0 + y)$ , we may assume that  $x_0 = 0$ . Assume that  $B(0, r) \subset U$ , fix  $x \in B(0, r)$ , and define

$$h: (-1 - \varepsilon, 1 + \varepsilon) \rightarrow \mathbb{R}, \quad h(t) := g(tx),$$

where  $\varepsilon > 0$  satisfies  $(1 + \varepsilon)|x| < r$ . Then  $h$  is a  $C^k$  function on  $(-1 - \varepsilon, 1 + \varepsilon)$ , and repeated use of the fundamental theorem of calculus gives

$$\begin{aligned} h(t) &= h(t) - h(0) + h(0) = h(0) + \int_0^t h'(s) ds \\ &= h(0) + h'(0)t + \int_0^t (h'(s) - h'(0)) ds = h(0) + h'(0)t + \int_0^t \int_0^s h''(u) du ds \\ &= h(0) + h'(0)t + h''(0) \frac{t^2}{2} + \int_0^t \int_0^s (h''(u) - h''(0)) du ds \\ &= \dots \\ (2.1) \quad &= \sum_{i=0}^k h^{(i)}(0) \frac{t^i}{i!} + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} (h^{(k)}(t_k) - h^{(k)}(0)) dt_k \cdots dt_1. \end{aligned}$$

Here we used that  $\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} dt_k \cdots dt_1 = \frac{t^k}{k!}$  (exercise).

Now, computation shows

$$h'(t) = \partial_j f(tx) x_j, \quad h''(t) = \partial_{j_l} f(tx) x_j x_l, \quad \dots$$

and

$$h^{(k)}(t) = \partial_{j_1 \dots j_k} f(tx) x_{j_1} \cdots x_{j_k}.$$

Applying (2.1) with  $t = 1$  gives the result in the theorem, where

$$R_k(x) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \left[ \partial_{j_1 \dots j_k} f(t_k x) - \partial_{j_1 \dots j_k} f(0) \right] x_{j_1} \cdots x_{j_k} dt_k \cdots dt_1.$$

The bound for  $R_k$  follows since  $\partial_{j_1 \dots j_k} f$  is uniformly continuous on compact sets.  $\square$

At this point it may be good to mention another convenient form of the Taylor expansion, which we state but will not use. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  be the set of natural numbers.

Then  $\mathbb{N}^n$  consists of all  $n$ -tuples  $\alpha = (\alpha_1, \dots, \alpha_n)$  where the  $\alpha_j$  are nonnegative integers. Such an  $n$ -tuple is called a multi-index. We write  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . For partial derivatives, the notation

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

will be used. We also use the notation  $\alpha! = \alpha_1! \dots \alpha_n!$ .

**Theorem 2.3** (Taylor expansion, multi-index version). Let  $f \in C^k(U)$ , let  $x_0 \in U$ , and assume that  $B(x_0, r) \subset U$ . If  $x \in B(x_0, r)$ , then

$$f(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha + R_k(x; x_0),$$

where  $R_k$  satisfies similar bounds as before.

*Proof.* Exercise. □

**2.2. Tensor fields.** If  $f \in C^k(U)$ , if  $x \in U$  and if  $v \in \mathbb{R}^n$  is such that  $|v|$  is sufficiently small, we write the Taylor expansion given in Theorem 2.1 in the form

$$f(x + v) = \sum_{l=0}^k \frac{1}{l!} \left[ \sum_{j_1, \dots, j_l=1}^n \partial_{j_1 \dots j_l} f(x) v_{j_1} \right] + R_k(x + v; x).$$

The first few terms are

$$f(x + v) = f(x) + \partial_j f(x) v_j + \frac{1}{2} \partial_{jk} f(x) v_j v_k + \dots$$

Looking at the terms of various degree motivates the following definition.

**Definition 2.4** (Tensor fields). An  $m$ -tensor field in  $U$  is a collection of functions  $u = (u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$ , where each  $u_{j_1 \dots j_m}$  is in  $C^\infty(U)$ . The tensor field  $u$  is called *symmetric* if  $u_{j_1 \dots j_m} = u_{j_{\sigma(1)} \dots j_{\sigma(m)}}$  for any  $j_1, \dots, j_m$  and for any  $\sigma$  which is a permutation of  $\{1, \dots, m\}$ .

**Remark 2.5.** This definition is specific to  $\mathbb{R}^n$ , since we are deliberately not allowing any other coordinate systems than the Cartesian one. Later on we will consider tensor fields on manifolds, and their transformation rules under coordinate changes will be an important feature (these will decide whether the tensor field is covariant, contravariant or mixed). However, upon fixing a local coordinate system, all tensor fields will look essentially like the ones defined above.

**Example 2.6.** (1) The 0-tensor fields in  $U$  are just the scalar functions  $u \in C^\infty(U)$   
 (2) The 1-tensor fields in  $U$  are of the form  $u = (u_j)_{j=1}^n$ , where  $u_j \in C^\infty(U)$ . Thus 1-tensor fields are exactly the vector fields in  $U$ ; the tensor  $(u_j)_{j=1}^n$  is identified with  $(u_1, \dots, u_n)$ .  
 (3) The 2-tensor fields in  $U$  are of the form  $u = (u_{j,k})_{j,k=1}^n$ , where  $u_{j,k} \in C^\infty(U)$ . Thus 2-tensor fields can be identified with smooth matrix functions in  $U$ . The 2-tensor field is symmetric if the matrix is symmetric.

- (4) If  $f \in C^\infty(U)$ , then we have for any  $m \geq 0$  an  $m$ -tensor field  $u = (\partial_{j_1 \dots j_m} f)_{j_1, \dots, j_m=1}^n$  consisting of partial derivatives of  $f$ . This tensor field is symmetric since the mixed partial derivatives can be taken in any order.

Again by looking at the terms in the Taylor expansion, one can also think that an  $m$ -tensor  $u = (u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$  acts on a vector  $v \in \mathbb{R}^n$  by the formula

$$v \mapsto u_{j_1 \dots j_m}(x) v^{j_1} \dots v^{j_m}.$$

The last expression can be interpreted as a multilinear map acting on the  $m$ -tuple of vectors  $(v, \dots, v)$ .

**Definition 2.7** (Multi-linear map). If  $m \geq 0$ , an  $m$ -linear map is any map

$$L: \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$$

such that  $L$  is linear in each of its variables separately.

The following theorem is almost trivial, but for later purposes it will be good to know that a tensor field can be thought of in two ways: either as a collection of coordinate functions, or as a map on  $U$  that takes values in the set of multilinear maps.

**Theorem 2.8** (Tensors as multilinear maps). If  $u = (u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$  is an  $m$ -tensor field on  $U \subset \mathbb{R}^n$ , then for any  $x \in U$ , there is an  $m$ -linear map  $u(x)$  defined via

$$u(x)(v_1, \dots, v_m) = u_{j_1 \dots j_m}(x) v_1^{j_1} \dots v_m^{j_m}, \quad v_1, \dots, v_m \in \mathbb{R}^n,$$

and it holds that  $u_{j_1 \dots j_m}(x) = u(x)(e_{j_1}, \dots, e_{j_m})$ . Conversely, if  $T$  is a function that assigns to each  $x \in U$  an  $m$ -linear map  $T(x)$ , and if the function  $u_{j_1 \dots j_m}: x \mapsto T(x)(e_{j_1}, \dots, e_{j_m})$  are in  $C^\infty(U)$  for each  $j_1, \dots, j_m$ , then  $(u_{j_1 \dots j_m})$  is an  $m$ -tensor field in  $U$ .

*Proof.* Exercise. □

To get a picture of what Theorem 2.8 really says, we consider the case of 2-tensors. In this case,  $u = (u_{jk})$  can be identified with matrix-valued functions  $A = (a_{jk})_{j,k=1}^n$  with  $a_{jk}(x) = u_{jk}(x)$ . For each  $x \in U$ , we may regard  $A(x)$  as a 2-linear map via

$$A(x)(v, w) = vA(x)w^T = \sum_{j,k=1}^n a_{jk}(x) v^j w^k.$$

It is clear that  $A(x)$  is linear in both  $v$  and  $w$ , since  $A(x)(av_1 + v_2, w) = (av_1 + v_2)A(x)w^T = aA(x)(v_1, w) + A(x)(v_2, w)$  and  $A(x)(v, bw_1 + w_2) = vA(x)(bw_1 + w_2)^T = bA(x)(v, w_1) + A(x)(v, w_2)$  hold for each  $a, b \in \mathbb{R}$ .

**2.3. Vector fields and differential forms.** Let  $U \subset \mathbb{R}^n$  be an open set. We wish to consider vector fields on  $U$  and certain operations related to vector fields.

**Definition 2.9** (Vector fields). A  $C^k$  vector field in  $U$  is a map  $F = (F_1, \dots, F_n): U \rightarrow \mathbb{R}^n$  such that all the component functions  $F_j$  are in  $C^k(U)$ . The set of vector fields on  $U$  is denoted by  $C^k(U, \mathbb{R}^n)$ .

Recall from Section 2.2 that vector fields are the same as 1-tensor fields. If  $u \in C^\infty(U)$ , the gradient of  $u$  gives rise to a vector field in  $U$ :

$$\text{grad}: C^\infty(U) \rightarrow C^\infty(U, \mathbb{R}^n), \quad \text{grad}(u) = (\partial_1 u, \dots, \partial_n u).$$

If  $F \in C^\infty(U, \mathbb{R}^n)$ , the *divergence* of  $F$  gives rise to a function in  $U$ :

$$\text{div}: C^\infty(U, \mathbb{R}^n) \rightarrow C^\infty(U), \quad \text{div}(F) = \partial_1 F_1 + \dots + \partial_n F_n.$$

The following basic identity suggests that in order to define the Laplace operator on a space, it may be enough to have a reasonable definition of divergence and gradient.

**Lemma 2.10.**  $\text{div} \circ \text{grad} = \Delta$ .

*Proof.*  $\text{div}(\text{grad}(u)) = \partial_1(\partial_1 u) + \dots + \partial_n(\partial_n u) = \Delta u.$  □

We will consider further operations on vector fields in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

**Curl in  $\mathbb{R}^2$ .** Let  $U \subset \mathbb{R}^2$  be open. If  $F \in C^\infty(U, \mathbb{R}^2)$ , the curl of  $F$  is the function

$$\text{curl}(F) := \partial_1 F_2 - \partial_2 F_1.$$

Thus  $\text{curl}: C^\infty(U, \mathbb{R}^2) \rightarrow C^\infty(U)$ .

**Curl in  $\mathbb{R}^3$ .** Let  $U \subset \mathbb{R}^3$  be open. If  $F \in C^\infty(U, \mathbb{R}^3)$ , the curl of  $F$  is the vector field

$$\text{curl}(F) := \nabla \times F = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1).$$

**Lemma 2.11.** In two dimensions, one has

$$\text{curl} \circ \text{grad} = 0.$$

In three dimensions, one has

$$\text{curl} \circ \text{grad} = 0, \quad \text{div} \circ \text{curl} = 0.$$

*Proof.* If  $U \subset \mathbb{R}^2$  and  $u \in C^\infty(U)$ , we have

$$\text{curl}(\text{grad}(u)) = \partial_1(\partial_2 u) - \partial_2(\partial_1 u) = 0.$$

If  $U \subset \mathbb{R}^3$  and  $u \in C^\infty(U)$ , we have

$$\text{curl}(\text{grad}(u)) = (\partial_2 \partial_3 u - \partial_3 \partial_2 u, \partial_3 \partial_1 u - \partial_1 \partial_3 u, \partial_1 \partial_2 u - \partial_2 \partial_1 u) = 0.$$

Moreover, for  $F \in C^\infty(U, \mathbb{R}^3)$  we have

$$\text{div}(\text{curl}(F)) = \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) = 0.$$

□

The previous lemma can be described in terms of two sequences: if  $U \subset \mathbb{R}^2$  consider

$$(2.2) \quad C^\infty(U) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(U)$$

and if  $U \subset \mathbb{R}^3$  consider

$$(2.3) \quad C^\infty(U) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U).$$

In both sequences, the composition of any two subsequent operators is zero. This suggests that there may be further structure which underlies these situations and might extend

to higher dimensions. This is indeed the case, and the calculus of differential forms (or exterior algebra) was developed to reveal this structure. We will next discuss this calculus in a simple case.

**Differential forms.** The purpose will be to rewrite for instance (2.3) as a sequence

$$(2.4) \quad \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U),$$

where  $\Omega^k(U)$  will be the set of differential  $k$ -forms on  $U \subset \mathbb{R}^3$ , and  $d$  will be a universal operator that reduces to grad, curl, and div in the respective degrees.

Let  $U \subset \mathbb{R}^n$  be open. Motivated by (2.2) and (2.3), we define

$$\Omega^0(U) := C^\infty(U)$$

and

$$\Omega^1(U) := C^\infty(U, \mathbb{R}^n).$$

Thus  $\Omega^0(U)$  is the set of smooth functions in  $U$ , and any  $\alpha \in \Omega^1(U)$  can be identified with a vector field  $\alpha = (\alpha_j)_{j=1}^n$ , where  $\alpha_j \in C^\infty(U)$ . We write formally

$$\alpha = (\alpha_j)_{j=1}^n = \alpha_j dx^j.$$

**Remark 2.12.** For the purposes of this section it is enough to think of  $dx^j$  as a formal object. However, the proper way to think of  $dx^j$  would be as a 1-form (the exterior derivative of the function  $x^j: U \rightarrow \mathbb{R}$ ), i.e. as a map that assigns to each  $x \in U$  the linear map  $dx^j|_x: T_x U \rightarrow \mathbb{R}$  that satisfies  $dx^j|_x(e_k) = \delta_k^j$ , where  $\{e_1, \dots, e_n\}$  is the standard basis of  $T_x U \approx \mathbb{R}^n$ .

To define  $\Omega^k(U)$  for  $k \geq 2$ , first define the set of ordered  $k$ -tuples

$$\mathcal{I}_k := \{(i_1, \dots, i_k) : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}.$$

If  $I \in \mathcal{I}_k$ , we consider the formal object

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}.$$

Then  $\Omega^k(U)$  will be thought of as the set

$$\Omega^k(U) = \{\alpha_I dx^I : \alpha_I \in C^\infty(U)\},$$

where the sum is over all  $I \in \mathcal{I}_k$ . The number of elements in  $\mathcal{I}_k$  is  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ . We can make the above formal definition rigorous.

**Definition 2.13** (Differential form). If  $U \subset \mathbb{R}^n$ , define for  $0 \leq k \leq n$

$$\Omega^k(U) := C^\infty\left(U, \mathbb{R}^{\binom{n}{k}}\right).$$

The elements of  $\Omega^k(U)$  are called *differential  $k$ -forms* on  $U$ , and any differential  $k$ -form  $\alpha \in \Omega^k(U)$  can be written as

$$\alpha = (\alpha_I)_{I \in \mathcal{I}_k} = \alpha_I dx^I,$$

where  $\alpha_I \in C^\infty(U)$  for each  $I$ .



**Remark 2.14.** Note that since  $\binom{n}{k} = \binom{n}{n-k}$ , the set  $\Omega^{n-1}(U)$  can be identified with the set of vector fields on  $U$ , and  $\Omega^n(U)$  with  $C^\infty(U)$ . In fact one has

$$\Omega^{n-1}(U) = \left\{ \sum_{j=1}^n \alpha_j dx^1 \wedge \cdots \wedge \hat{dx}^j \wedge \cdots \wedge dx^n; \alpha_j \in C^\infty(U) \right\}$$

$$\Omega^n(U) = \left\{ f dx^1 \wedge \cdots \wedge dx^n; f \in C^\infty(U) \right\},$$

where  $\hat{dx}^j$  means that  $dx^j$  is omitted from the wedge product.

The above definition is correct, but to keep things simple we have avoided a detailed discussion of the *wedge product*  $\wedge$ . To define the  $d$  operator in (2.4) properly we need to say a little bit more. The wedge product is an associative product on elements of the form  $dx^I$ , satisfying

$$dx^j \wedge dx^k = -dx^k \wedge dx^j,$$

and more generally if  $J = (j_1, \dots, j_k)$  is a  $k$ -tuple, with  $j_1, \dots, j_k \in \{1, \dots, n\}$  (not necessarily ordered), we should have

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k} = (-1)^{\text{Sign}(\sigma)} dx^{j_{\sigma(1)}} \wedge \cdots \wedge dx^{j_{\sigma(k)}},$$

where  $\sigma$  is any permutation of  $\{1, \dots, k\}$ . This implies two conditions:

- $dx^{j_1} \wedge \cdots \wedge dx^{j_k} = 0$  if  $(j_1, \dots, j_k)$  contains a repeated index
- $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$  can be expressed as  $\pm dx^I$  for a unique  $I \in \mathcal{I}_k$  if  $(j_1, \dots, j_k)$  contains no repeated index.

With this understanding we make the following definition.

**Definition 2.15** (Exterior derivative). The *exterior derivative* is the map  $d: \Omega^k(U) \rightarrow \Omega^{k+1}(U)$  defined by

$$d(\alpha_I dx^I) := \partial_j \alpha_I dx^j \wedge dx^I.$$

**Example 2.16.** (1) If  $f \in \Omega^0(U)$  (so  $f \in C^\infty(U)$ ), then  $df$  is the differential of  $f$  written as a 1-form:

$$df = \partial_j f dx^j.$$

(2) If  $\alpha \in \Omega^1(U)$ , say  $\alpha = \alpha_k dx^k$  for some  $\alpha_k \in C^\infty(U)$ , then

$$d\alpha = \partial_j \alpha_k dx^j \wedge dx^k = \sum_{1 \leq j < k \leq n} (\partial_j \alpha_k - \partial_k \alpha_j) dx^j \wedge dx^k.$$

(3) Any  $u \in \Omega^n(U)$  satisfies  $du = 0$  since  $dx^{j_1} \wedge \cdots \wedge dx^{j_{n+1}} = 0$  whenever  $j_1, \dots, j_{n+1} \in \{1, \dots, n\}$  and there will be a repeated index.

The second example above gives an  $n$ -dimensional analogue of the curl operator, as also suggested by the following lemma:

**Lemma 2.17** (The exterior derivatives in two and three dimensions). (1) Let  $U \subset \mathbb{R}^2$ . If  $f \in \Omega^0(U)$ , then

$$df = (\text{grad}(f))_j dx^j.$$

If  $\alpha = F_1 dx^1 + F_2 dx^2 \in \Omega^1(U)$  and  $F = (F_1, F_2)$ , then

$$d\alpha = (\operatorname{curl}(F)) dx^1 \wedge dx^2.$$

(2) Let  $U \subset \mathbb{R}^3$ . If  $f \in \Omega^0(U)$ , then

$$df = (\operatorname{grad}(f))_j dx^j.$$

If  $\alpha = F_j dx^j \in \Omega^1(U)$  and  $F = (F_1, F_2, F_3)$ , then

$$d\alpha = (\operatorname{curl}(F))_j dx^{\hat{j}},$$

where

$$dx^{\hat{1}} := dx^2 \wedge dx^3, dx^{\hat{2}} := dx^3 \wedge dx^1, \text{ and } dx^{\hat{3}} := dx^1 \wedge dx^2.$$

Finally, if  $u = F_j dx^{\hat{j}} \in \Omega^2(U)$  and  $F = (F_1, F_2, F_3)$ , then

$$du = ((\operatorname{div}(F))) dx^1 \wedge dx^2 \wedge dx^3.$$

*Proof.* Exercise. □

Let us now verify that  $d \circ d$  is always zero.

**Lemma 2.18.**  $d \circ d = 0$  on  $\Omega^k(U)$  for any  $k$  with  $0 \leq k \leq n$ .

*Proof.* If  $\alpha = \alpha_I dx^I \in \Omega^k(U)$ , then

$$d\alpha = \sum_{k=1}^n \sum_{I \in \mathcal{I}_k} \partial_k \alpha_I dx^k \wedge dx^I$$

and

$$d(d\alpha) = \sum_{j,k=1}^n \sum_{I \in \mathcal{I}_k} \partial_{jk} \alpha_I dx^j \wedge dx^k \wedge dx^I.$$

By the properties of the wedge product, we get

$$d(d\alpha) = \sum_{1 \leq j < k \leq n} \sum_{I \in \mathcal{I}_k} (\partial_{jk} \alpha_I - \partial_{kj} \alpha_I) dx^j \wedge dx^k \wedge dx^I,$$

which is zero since the mixed partial derivatives are equal. □

If  $U \subset \mathbb{R}^n$  is open, we therefore have a sequence

$$(2.5) \quad \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U)$$

and the composition of any two subsequent operators is zero. This gives the desired generalization of (2.2) and (2.3) to any dimension. In fact we have obtained much more: as we will see during this course, differential forms turn out to be an object of central importance in many kinds of analysis on manifolds.

**Differential forms as tensors.** It will be useful to interpret differential forms as tensor fields satisfying an extra condition.

**Definition 2.19** (Alternating tensor field). An  $m$ -tensor field  $(u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$  in  $U \subset \mathbb{R}^n$  is called *alternating* if  $u_{j_{\sigma(1)} \dots j_{\sigma(m)}} = (-1)^{\operatorname{Sign}(\sigma)} u_{j_1 \dots j_m}$  for any  $j_1, \dots, j_m$  and for any  $\sigma$  which is a permutation of  $\{1, \dots, m\}$ .

We understand that 0-tensor fields and 1-tensor fields are always alternating. A 2-tensor field  $u = (u_{jk})_{j,k=1}^n$  is alternating if and only if  $u_{kj} = -u_{jk}$  for any  $j, k$ , i.e. the matrix  $(u_{jk})$  is skew-symmetric at each point. An  $m$ -tensor field  $u = (u_{j_1 \dots j_m})$  is alternating if and only if  $u_{j_1 \dots j_m}$  changes sign when any two indices are interchanged (since any permutation can be expressed as the product of transpositions). Note that for an alternating tensor,  $u_{j_1 \dots j_m} = 0$  whenever  $(j_1, \dots, j_m)$  contains a repeated index.

**Theorem 2.20.** If  $U \subset \mathbb{R}^n$  is open and  $0 \leq k \leq n$ , the set  $\Omega^k(U)$  can be identified with the set of alternating  $k$ -tensor fields on  $U$ .

*Proof.* Consider the map

$$T: \Omega^k(U) \rightarrow \{\text{alternating } k\text{-tensors}\}, \quad \alpha dx^I \mapsto (\tilde{\alpha}_{j_1 \dots j_k}),$$

where

$$\tilde{\alpha}_{j_1 \dots j_k} := \begin{cases} 0, & (j_1, \dots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}} (-1)^{\text{Sign}(\sigma)} \alpha_I, & (j_1, \dots, j_k) \text{ contains no repeated index.} \end{cases}$$

Here,  $\sigma$  is the permutation of  $\{1, \dots, k\}$  such that  $I = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$  is the unique element of  $\mathcal{I}_k$  containing the same entries as  $(j_1, \dots, j_k)$ . The constant  $\frac{1}{\sqrt{k!}}$  is a harmless normalizing factor which will be useful later. Then  $\tilde{\alpha}_{j_1 \dots j_k}$  is alternating by construction. It is clear that  $T$  is injective, and surjectivity follows since any alternating tensor is uniquely determined by the elements  $\tilde{\alpha}_I$  where  $I \in \mathcal{I}_k$ .  $\square$

**Cohomology.** By Lemma (2.18), we observe that

$$u = d\alpha \text{ for some } \alpha \in \Omega^{k-1}(U) \Rightarrow du = 0.$$

This may be rephrased as follows:

$$\text{Im}(d|_{\Omega^{k-1}(U)}) \text{ is a linear subspace of } \ker(d|_{\Omega^k(U)}).$$

We express this in one more way: if  $u \in \Omega^k(U)$ , we say that  $u$  is closed if  $du = 0$  and that  $u$  is exact if  $u = d\alpha$  for some  $\alpha \in \Omega^{k-1}(U)$ . Thus, any exact differential form is closed. The question of whether any closed form is exact depends on the topological properties of  $U$ . To study this property we make the following definition.

**Definition 2.21** (de Rham cohomology). The de Rham cohomology groups of  $U$  are defined by

$$H_{\text{dR}}^k(U) = \ker(d|_{\Omega^k(U)}) / \text{Im}(d|_{\Omega^{k-1}(U)}), \quad 0 \leq k \leq n.$$

By this definition each  $H_{\text{dR}}^k(U)$  is in fact a (quotient) vector space, not just a group. Recall that

Any closed  $k$ -form is exact if and only if  $H_{\text{dR}}^k(U) = \{0\}$ . This happens for all  $k \geq 1$  at least when  $U$  has very simple topology.

**Lemma 2.22** (Poincaré lemma). If  $U \subset \mathbb{R}^n$  is open and star-shaped with respect to some  $x_0 \in U$  (meaning that for any  $x \in U$  the line segment between  $x_0$  and  $x$  lies in  $U$ ), then

$$H_{\text{dR}}^k(U) = \begin{cases} \mathbb{R}, & k = 0, \\ \{0\}, & 1 \leq k \leq n. \end{cases}$$

*Proof.* For simplicity we only do the proof for  $n = 2$ , see [8] for the general case (which is somewhat more involved). Assume that  $U$  is star-shaped with respect to 0. We have

$$H_{\text{dR}}^0(U) = \ker(d|_{\Omega^0(U)}) = \{f \in C^\infty(U), \text{grad}(f) = 0\}.$$

Since  $U$  is connected and star-shaped with respect to 0,  $\nabla f = 0$  on  $U$  implies that  $f \equiv f(0)$  is constant. Thus  $H_{\text{dR}}^0(U)$  is one-dimensional and isomorphic to  $\mathbb{R}$ .

We next show that  $H_{\text{dR}}^1(U) = \{0\}$ , that is, for any  $F \in C^\infty(U, \mathbb{R}^2)$ , we have

$$\text{curl}(F) = 0 \Rightarrow F = \text{grad}(f) \text{ for some } f \in C^\infty(U).$$

Let  $F = (F_1, F_2)$  satisfy  $\partial_1 F_2 - \partial_2 F_1 = 0$ . Then  $f$  should be some kind of integral of  $F$ , in fact we may just take

$$f(x) := \int_0^1 F_j(tx)x^j dt, \quad x \in U.$$

Since  $\partial_1 F_2 = \partial_2 F_1$ , we have

$$\begin{aligned} \partial_1 f(x) &= \int_0^1 [\partial_1 F_j(tx)tx^j + F_1(tx)] dt \\ &= \int_0^1 [\partial_1 F_1(tx)tx^1 + \partial_2 F_1(tx)tx^2 + F_1(tx)] dt \\ &= \int_0^1 \frac{d}{dt} [tF_1(tx)] dt = F_1(x). \end{aligned}$$

Similarly,  $\partial_2 f(x) = F_2(x)$ , showing that  $F = \text{grad}(f)$ .

Finally, we show that  $H_{\text{dR}}^2(U) = \{0\}$ , which means that

$$f \in C^\infty(U) \Rightarrow f = \text{curl}(F) \text{ for some } F \in C^\infty(U, \mathbb{R}^2).$$

As in the previous case,  $F_j$  should be integrals of  $f$ . We may define

$$F_1(x) := - \int_0^1 f(tx)tx_2 dt \quad \text{and} \quad F_2(x) := \int_0^1 f(tx)tx_1 dx.$$

Then

$$\begin{aligned} \partial_1 F_2 - \partial_2 F_1 &= \int_0^1 [\partial_1 f(tx)t^2x_1 + \partial_2 f(tx)t^2x_2 + 2tf(tx)] dt \\ &= \int_0^1 \frac{d}{dt} [t^2 f(tx)] dt = f(x). \end{aligned}$$

□

We conclude by mentioning some facts about the de Rham cohomology groups (for more details see [8]):

- The de Rham cohomology groups are *topological invariants*: if  $U$  and  $V$  are homeomorphic open sets in Euclidean space, then  $H_{\text{dR}}^k(U)$  and  $H_{\text{dR}}^k(V)$  are isomorphic as vector spaces for each  $k$ . This gives a potential way of showing that two sets  $U$  and  $V$  are not homeomorphic; it would be enough to check that some cohomology groups are not isomorphic
- Note however that it is possible for non-homeomorphic spaces to have the same cohomology groups
- In many cases (e.g. if  $U \subset \mathbb{R}^n$  is a bounded open set with nice boundary), the vector spaces  $H_{\text{dR}}^k(U)$  are finite dimensional. The dimension of  $H_{\text{dR}}^k(U)$  is a known topological invariant, namely the  $k$ -th *Betti number* of  $U$ .
- Very loosely speaking, the cohomology groups may give some information about “holes” in a set. For instance, if  $K_1, \dots, K_N$  are disjoint closed balls in  $\mathbb{R}^n$ , then

$$H_{\text{dR}}^k(\mathbb{R}^n \setminus \cup_j^N K_j) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ \mathbb{R}^N, & \text{if } k = n - 1, \\ \{0\}, & \text{otherwise} \end{cases}$$

Later in this course we will discuss Hodge theory, which studies the cohomology groups  $H_{\text{dR}}^k(M)$  where  $M$  is a compact manifold via the Laplace operator acting on differential forms on  $M$ .

**2.4. Riemannian metrics.** An open set  $U \subset \mathbb{R}^n$  is often thought to be “homogeneous” (the set looks the same near every point) and “flat” (if  $U$  is considered as a subset of  $\mathbb{R}^{n+1}$  lying in the hyperplane  $\{x_{n+1} = 0\}$ , then  $U$  has the geometry induced by the flat hypersurface  $\{x_{n+1} = 0\}$ ). In this section, we will introduce extra structure on  $U$  which makes it “inhomogeneous” (the properties of the set vary from point to point) and “curved” ( $U$  has some geometry that is different from the geometry induced by a flat hypersurface  $\{x_{n+1} = 0\}$ ).

**Motivation.** An intuitive way of introducing this extra structure is to think of  $U$  as a medium where sound waves propagate. The properties of the medium are described by a function  $c: U \rightarrow \mathbb{R}_+$ , which is thought of as the sound speed of the medium. If  $U$  is homogeneous, the sound speed is constant ( $c(x) = 1$  for each  $x \in U$ ), but if  $U$  is inhomogeneous, then the sound speed varies from point to point.

Consider now a  $C^1$  curve  $\gamma: [0, 1] \rightarrow U$ . The tangent vector  $\dot{\gamma}(t)$  of this curve is thought to be a vector at the point  $\gamma(t)$ . If the sound speed is constant ( $c \equiv 1$ ), the length of the tangent vector is just the Euclidean length:

$$\left| \dot{\gamma}(t) \right|_e := \left[ \sum_{j=1}^n \dot{\gamma}^j(t)^2 \right]^{\frac{1}{2}}.$$

In case of a general sound speed  $c: U \rightarrow \mathbb{R}_+$ , one can think that at points where  $c$  is large the curve moves very quickly and consequently has short length. Thus we may define the

length of  $\dot{\gamma}(t)$  with respect to the sound speed  $c$  by

$$\left| \dot{\gamma}(t) \right|_c := \frac{1}{c(\gamma(t))} \left[ \sum_{j=1}^n \dot{\gamma}^j(t)^2 \right]^{\frac{1}{2}}.$$

It is useful to generalize the above setup in two directions. First, in addition to measuring lengths of tangent vectors we would also like to measure angles between tangent vectors (in particular we want to know when two tangent vectors are orthogonal). Second, if the sound speed is a scalar function on  $U$ , then the length of a tangent vector is independent of its direction (the medium is *isotropic*). We wish to allow the medium to be *anisotropic*, which will mean that the sound speed may depend on direction and should be a matrix valued function.

In order to measure lengths and angles of tangent vectors, it is enough to introduce an inner product on the space of tangent vectors at each point. The tangent space is defined as follows:

**Definition 2.23** (Tangent space). If  $U \subset \mathbb{R}^n$  is open and  $x \in U$ , the *tangent space* at  $x$  is defined as

$$T_x U := \{x\} \times \mathbb{R}^n.$$

The *tangent bundle* of  $U$  is the set

$$TU := \bigcup_{x \in U} T_x U.$$

Of course, each  $T_x U$  can be identified with  $\mathbb{R}^n$  (and we will often do so), and a vector  $v \in T_x U$  is written in terms of its coordinates as  $v = (v^1, \dots, v^n)$ . Now if  $\langle \cdot, \cdot \rangle$  is any inner product on  $\mathbb{R}^n$ , there is some positive definite symmetric matrix  $A = (a_{jk})_{j,k=1}^n$  such that

$$\langle v, w \rangle = Av \cdot w, \quad v, w \in \mathbb{R}^n.$$

(The proof is left as an exercise, hint: take  $a_{jk} = \langle e_j, e_k \rangle$ ) The next definition introduces an inner product on the space of tangent vectors at each point:

**Definition 2.24** (Riemannian metric). A *Riemannian metric* on  $U$  is a matrix-valued function  $g = (g_{jk})_{j,k=1}^n$  such that each  $g_{jk}$  is in  $C^\infty(U)$ , and  $(g_{jk}(x))$  is a positive definite symmetric matrix for each  $x \in U$ . The corresponding *inner product* on  $T_x U$  is defined by

$$\langle v, w \rangle_g := g_{jk}(x)v^j w^k, \quad v, w \in T_x U.$$

The *length* of a tangent vector is

$$|v|_g := \langle v, v \rangle_g^{1/2} = (g_{jk}(x)v^j v^k)^{1/2}, \quad v \in T_x U.$$

The *angle* between two tangent vectors  $v, w \in T_x U$  is the number  $\theta_g(v, w) \in [0, \pi]$  defined by

$$\cos \theta_g(v, w) = \frac{\langle v, w \rangle_g}{|v|_g |w|_g}.$$

We will often drop the subscript and write  $\langle \cdot, \cdot \rangle$  or  $|\cdot|$  if the metric  $g$  is fixed. To connect the above definition to the discussion about sound speeds, a scalar sound speed

$c(x)$  corresponds to the Riemannian metric

$$g_{jk}(x) = \frac{1}{c(x)^2} \delta_{jk}.$$

Finally, we introduce some notation that will be very useful.

**Notation.** If  $g = (g_{jk})$  is a Riemannian metric on  $U$ , we write

$$(g^{jk})_{j,k=1}^n = g^{-1}$$

for the inverse matrix of  $(g_{jk})_{j,k=1}^n$ , and

$$|g| = \det(g)$$

for the determinant of the matrix  $(g_{jk})_{j,k=1}^n$ . In particular, we note that  $g_{jk}g^{kl} = \delta_j^l$  for any  $j, l = 1, \dots, n$ .

**2.5. Geodesics. Lengths of curves.** Consider an open set  $U$  that is equipped with a Riemannian metric  $g$ . As we saw above, one can measure lengths of tangent vectors with respect to  $g$ , and this makes it possible to measure lengths of curves as well.

**Definition 2.25** (Regular curve and its length). A smooth map  $\gamma: [a, b] \rightarrow U$  whose tangent vector  $\dot{\gamma}(t)$  is always nonzero is called a *regular curve*. The *length* of  $\gamma$  is defined by

$$L_g(\gamma) := \int_a^b |\dot{\gamma}(t)|_g dt.$$

The length of a piecewise regular curve is defined as the sum of lengths of the regular parts.

The *Riemannian distance* between two points  $p, q \in U$  is defined by

$$d_g(p, q) := \inf \left\{ L_g(\gamma); \gamma: [a, b] \rightarrow U \text{ is piecewise regular with } \gamma(a) = p \text{ and } \gamma(b) = q \right\}.$$

Since we only use the given Riemannian metric  $g$  on  $U$ , we will often omit the sub/sup-script  $g$  in the corresponding quantity.

**Fact.**  $L(\gamma)$  is independent of the way the curve  $\gamma$  is parametrized, and that we may always parametrize  $\gamma$  by *arc-length* so that  $|\dot{\gamma}(t)| = 1$  for all  $t$ . (Proof is left as an exercise)

The previous exercise shows that we can always reparametrize a piecewise regular curve  $\gamma$  by arc length, so that one will have  $|\dot{\gamma}(t)| = 1$  for all  $t$ . A curve satisfying  $|\dot{\gamma}(t)| \equiv 1$  is called a *unit speed curve* (similarly a curve satisfying  $|\dot{\gamma}(t)| \equiv \text{constant}$  is called a *constant speed curve*).

**Geodesic equation.** We now wish to show that any length minimizing curve satisfies a certain ordinary differential equation.

**Theorem 2.26** (Length minimizing curves are geodesics). Suppose  $U \subset \mathbb{R}^n$  is open, let  $g$  be a Riemannian metric on  $U$ , and let  $\gamma: [a, b] \rightarrow U$  be a piecewise regular unit speed curve. Assume that  $\gamma$  minimizes the distance between its endpoints, in the sense that

$$L(\gamma) \leq L(\eta)$$

for any piecewise regular curve  $\eta$  from  $\gamma(a)$  to  $\gamma(b)$ . Then  $\gamma$  is a regular curve, and it satisfies the geodesic equation

$$(2.6) \quad \ddot{\gamma}^l(t) + \Gamma_{jk}^l(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0, \quad 1 \leq l \leq n,$$

where  $\Gamma_{jk}^l$  are the Christoffel symbols of the metric  $g$ :

$$\Gamma_{jk}^l := \frac{1}{2}g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}), \quad 1 \leq j, k, l \leq n.$$

**Example 2.27.** If  $g$  is the Euclidean metric on  $U$ , so that  $g_{jk}(x) = \delta_{jk}$ , then all the Christoffel symbols  $\Gamma_{jk}^l$  are zero. The geodesic equation becomes just

$$\ddot{\gamma}^l(t) = 0, \quad 1 \leq l \leq n.$$

Solving this equation shows that

$$\gamma(t) = tv + w$$

for some vectors  $v, w \in \mathbb{R}^n$ . Thus Theorem 2.26 recovers the classical fact that any length minimizing curve in Euclidean space is a line segment.

Any smooth curve that satisfies the geodesic equation (2.6) is called a *geodesic*, and the conclusion of Theorem 2.26 can be rephrased so that any length minimizing curve is a geodesic. The fact that length minimizing curves satisfy the geodesic equation gives powerful tools for studying these curves. For instance, one can show that

- any geodesic has constant speed and is therefore regular
- given any  $x \in U$  and  $v \in T_x U$ , there is a unique geodesic starting at point  $x$  in direction  $v$
- any geodesic minimizes length at least locally (but not always globally)
- a set  $U$  with Riemannian metric  $g$  is *geodesically complete*, meaning that every geodesic is defined for all  $t \in \mathbb{R}$ , if and only if the metric space  $(U, d_g)$  is complete (this is the *Hopf-Rinow theorem*).

**Variations of curves.** Let  $\gamma: [a, b] \rightarrow U$  be a piecewise regular length minimizing curve. We will prove Theorem 2.26 by considering families of curves  $(\gamma_s)$  where  $s \in (-\varepsilon, \varepsilon)$  and  $\gamma_0 = \gamma$ , and all curves  $\gamma_s$  start at  $\gamma(a)$  and end at  $\gamma(b)$ . Such a family is called a *variation* (or a *fixed-endpoint variation*) of  $\gamma$ . By the length minimizing property,

$$L(\gamma_0) \leq L(\gamma_s) \quad \text{for } s \in (-\varepsilon, \varepsilon),$$

so if the dependence on  $s$  is at least  $C^1$  we obtain that  $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$ . This fact, applied to many different families  $\gamma_s$ , will imply that  $\gamma$  is smooth and solves the geodesic equation.

If  $(\gamma_s)$  is a family of curves with  $\gamma_0 = \gamma$ , we think of  $V(t) := \frac{\partial}{\partial s}\gamma_s(t)|_{s=0}$  as the “infinitesimal variation” of the curve  $\gamma$  that leads to the family  $(\gamma_s)$ . The vector  $V(t)$  should be thought of as an element of  $T_{\gamma(t)}U$ . The next result shows that one can reverse this process, and obtain a variation of  $\gamma$  from any given infinitesimal variation  $V$ .

In this result and below, we assume that the piecewise regular curve  $\gamma$  is fixed and that there is a subdivision of  $[a, b]$ ,

$$a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b,$$



such that the curves  $\gamma|_{(t_j, t_{j+1})}$  is regular for each  $j$  with  $0 \leq j \leq N$ .

**Lemma 2.28** (Variations of curves). If  $V: [a, b] \rightarrow \mathbb{R}^n$  is a continuous map such that  $V|_{(t_j, t_{j+1})}$  is  $C^\infty$  for each  $j$  and  $V(a) = V(b) = 0$ , then there exists  $\varepsilon > 0$  and a continuous map

$$\Gamma: (-\varepsilon, \varepsilon) \times [a, b] \rightarrow U$$

such that the curves  $\gamma_s: [a, b] \rightarrow U$ ,  $\gamma_s(t) := \Gamma(s, t)$  satisfying the following

- each  $\gamma_s$  is a piecewise regular curve with endpoints  $\gamma(a)$  and  $\gamma(b)$ , and  $\gamma_s|_{(t_j, t_{j+1})}$  is regular for each  $j$ ,
- $\gamma_0 = \gamma$ ,
- $s \mapsto \gamma_s(t)$  is  $C^\infty$  and  $\frac{d}{ds}\gamma_s(t)|_{s=0} = V(t)$  for each  $t \in [a, b]$ .

*Proof.* Define

$$\Gamma: (-\varepsilon, \varepsilon) \times [a, b] \rightarrow U, \quad \Gamma(s, t) := \gamma(t) + sV(t),$$

where  $\varepsilon$  is so small that  $\Gamma$  takes values in  $U$ . The properties follow immediately from the definition.  $\square$

We can now compute the derivative  $\frac{d}{ds}L(\gamma_s)|_{s=0}$  that was mentioned above. In classical terminology, this is called the *first variation* of the length functional.

**Lemma 2.29** (First variation formula). Let  $\gamma$  be a piecewise regular unit speed curve, and let  $(\gamma_s)$  be a variation of  $\gamma$  associated with  $V$  as in Lemma 2.28. Then

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = - \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \langle D_t \dot{\gamma}(t), V(t) \rangle dt - \sum_{j=1}^N \langle \Delta \dot{\gamma}(t_j), V(t_j) \rangle,$$

where  $D_t \dot{\gamma}(t)$  is the element of  $T_{\gamma(t)}U$  defined by

$$(D_t \dot{\gamma}(t))^l := \ddot{\gamma}^l(t) + \Gamma_{jk}^l(\gamma(t)) \dot{\gamma}^j(t) \dot{\gamma}^k(t), \quad 1 \leq l \leq n,$$

and  $\Delta \dot{\gamma}(t_j) := \dot{\gamma}(t_{j+}) - \dot{\gamma}(t_{j-})$  is the jump of  $\dot{\gamma}(t)$  at  $t_j$ .

**Remark 2.30.** We will later give an invariant meaning to  $D_t \dot{\gamma}(t)$  and interpret it as the covariant derivative of  $\dot{\gamma}(t)$  along the curve  $\gamma$ . However, at this point it is enough to think of  $D_t \dot{\gamma}(t)$  just as some expression that comes out when we compute the derivative  $\frac{d}{ds}L(\gamma_s)|_{s=0}$ .

*Proof.* Define

$$I(s) := L(\gamma_s) = \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \left[ g_{pq}(\gamma_s(t)) \dot{\gamma}_s^p(t) \dot{\gamma}_s^q(t) \right]^{\frac{1}{2}} dt.$$

To prepare for computing the derivative  $I'(0)$ , define two vector fields

$$T(t) := \partial_t \gamma_s(t)|_{s=0} = \dot{\gamma}(t), \quad V(t) := \partial_s \gamma_s(t)|_{s=0}.$$

Since  $|\dot{\gamma}_0(t)| = |T(t)| \equiv 1$  and  $(g_{jk})$  is symmetric, we have

$$I'(0) = \frac{1}{2} \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \left( \partial_r g_{pq}(\gamma(t)) V^r(t) T^p(t) T^q(t) + 2g_{pq}(\gamma(t)) \dot{V}^p(t) T^q(t) \right) dt.$$

Integrating by parts in the last term, this shows that

$$I'(0) = \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \left[ \frac{1}{2} \partial_r g_{pq}(\gamma) T^p T^q - \partial_m g_{rq}(\gamma) \dot{T}^q \right] V^r dt \\ + \sum_{j=0}^N \left[ \langle V(t_{j+1}), T(t_{j+1}) \rangle - \langle V(t_j), T(t_j) \rangle \right].$$

Using that  $V(t_0) = V(t_{N+1}) = 0$  and that  $V$  is continuous, the boundary term becomes  $-\sum_{j=1}^N \langle \Delta \dot{\gamma}(t_j), V(t_j) \rangle$  as required. For the integrals, we use that

$$\partial_m g_{rq}(\gamma) T^m T^q = \frac{1}{2} \left( \partial_m g_{rq}(\gamma) + \partial_q g_{rm}(\gamma) \right) T^m T^q,$$

which gives

$$-\langle D_t \dot{\gamma}(t), V(t) \rangle = -g_{rq}(\gamma) (\dot{T}^q + \Gamma_{jk}^q T^j T^k) V^r \\ = -g_{rq}(\gamma) \left( \dot{T}^q - \frac{1}{2} [\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}] T^j T^k \right) V^r \\ = -g_{rq}(\gamma) \left( \dot{T}^q + \frac{1}{2} \partial_r g_{pq} T^p T^q - \partial_m g_{rq} T^m T^q \right) V^r.$$

This completes the proof.  $\square$

*Proof of Theorem 2.26.* Let  $\gamma: [a, b] \rightarrow U$  be a piecewise regular unit speed curve that minimizes the length between its endpoints. If  $V$  is any vector field as in Lemma 2.28 and  $(\gamma_s)$  is the corresponding variation of  $\gamma$ , we must have

$$L(\gamma_0) \leq L(\gamma_s)$$

for  $s \in (-\varepsilon, \varepsilon)$ . Therefore,  $\frac{d}{ds} L(\gamma_s)|_{s=0} = 0$ . The first variation formula, Lemma 2.29, then shows that

$$(2.7) \quad \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \langle D_t \dot{\gamma}(t), V(t) \rangle dt + \sum_{j=1}^N \langle \Delta \dot{\gamma}(t_j), V(t_j) \rangle = 0$$

for any such  $V$ .

We first show that  $\gamma$  solves the geodesic equation on each interval  $(t_j, t_{j+1})$ . Fix  $j \in \{0, \dots, N\}$  and choose  $V$  such that

$$V(t) := \varphi(t) D_t \dot{\gamma}(t),$$

where  $\varphi$  is any function in  $C_0^\infty((t_j, t_{j+1}))$ . This  $V$  is an admissible choice in Lemma 2.29 and (2.7) implies that  $\square$

The previous proof shows actually more than stated in the theorem. We say that a piecewise regular curve  $\gamma$  is a *critical point* of the length functional  $L$  if  $\frac{d}{ds} L(\gamma_s)|_{s=0} = 0$  for any fixed-endpoint variation of  $\gamma$  as in Lemma 2.28.

**Theorem 2.31.** The critical points of  $L$  are exactly the geodesic curves.

*Proof.* The proof of Theorem 2.26 shows that any critical point of  $L$  is a geodesic curve. To see the converse, let  $\gamma$  be a geodesic curve so that  $\gamma$  is  $C^\infty$  and  $D_t \dot{\gamma}(t) = 0$  in  $[a, b]$ .

By the first variation formula, Lemma 2.29, any such curve satisfies  $\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = 0$ , so any geodesic must be a critical point of  $L$ .  $\square$

**Remark 2.32.** Let us give a more geometric interpretation of the proof of Theorem 2.26. Suppose that  $\gamma$  is a piecewise regular curve which is smooth in  $(t_j, t_{j+1})$  for  $0 \leq j \leq N$ . The preceding proof shows that

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = - \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \langle D_t \dot{\gamma}(t), V(t) \rangle dt - \sum_{j=1}^N \langle \Delta \dot{\gamma}(t_j), V(t_j) \rangle,$$

where  $(\gamma_s)$  is a variation of  $\gamma$  related to  $V$  as in Lemma 2.29. Choosing

$$V(t) := \varphi(t) D_t \dot{\gamma}(t),$$

where  $\varphi$  is a nonnegative function supported in  $(t_j, t_{j+1})$  shows that

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = - \int_{t_j}^{t_{j+1}} \varphi(t) |D_t \dot{\gamma}(t)|^2 dt \leq 0.$$

Thus if  $D_t \dot{\gamma}(t) \neq 0$  somewhere in  $(t_j, t_{j+1})$ , the derivative can be made strictly negative. This means we can always make the curve  $\gamma$  shorter by deforming it in the direction of  $D_t \dot{\gamma}(t)$ .

Assume now that  $\gamma$  solves the geodesic equation (2.6) in each segment  $(t_j, t_{j+1})$  where it is smooth. If one has  $\Delta \dot{\gamma}(t_j) \neq 0$  and if we choose  $V$  so that  $V(t_j) = \Delta \dot{\gamma}(t_j)$  and  $V(t_k) = 0$  for  $k \neq j$ , then

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = -|\Delta \dot{\gamma}(t_j)| < 0.$$

This shows that a “broken geodesic” with corner at  $t_j$  can always be made shorter by deforming it in the direction of  $\Delta \dot{\gamma}(t_j)$ . This argument of “rounding the corner” was the key point in showing that length minimizing curves are  $C^\infty$ .

**2.6. Integration and inner products.** This section will largely consist of definitions. We explain a natural way of integrating functions with respect to a Riemannian metric  $g$ , given by the volume form  $dV_g$ . This leads to an  $L^2$  inner product first for scalar functions and then for vector fields and tensor fields. Finally we discuss the codifferential operator  $\delta$ , which is the adjoint of the exterior derivative of  $d$  with respect to the  $L^2$  inner product on differential forms. On 1-forms  $\delta$  can be interpreted as a Riemannian divergence operator. The operator  $\delta$  will be used in the next section to define the Laplace operator.

**Integration.** Let  $U$  be an open set, and let  $g$  be a Riemannian metric on  $U$ . If  $f$  is a function in (say)  $C_c(U)$ , we wish to consider the integral of  $f$  over  $U$  with respect to the metric  $g$ . The idea is that the metric  $g$  gives a way of measuring infinitesimal volumes, in the same way that it allows to measure lengths and angles of tangent vectors.

**Motivation.** Since in this chapter we are restricting ourselves to using Cartesian coordinates, the integral of  $f$  over  $U$  should be approximately given by

$$(2.8) \quad \int_U f(x) d\text{Vol}_g \approx \sum_{j=1}^N f(x_j) \text{Vol}_g(Q_j),$$

where  $\{Q_1, \dots, Q_N\}$  are very small congruent cubes whose sides are parallel to the Cartesian coordinate axes such that the cubes approximately tile  $U$ , and  $x_j$  is the center of  $Q_j$ . Now if  $Q_j$  has sidelength  $h$ , one should have

$$\text{Vol}_g(Q_j) = \text{Vol}_g(h e_1|_{x_j}, \dots, h e_n|_{x_j}),$$

where  $\text{Vol}_g(v_1, \dots, v_n)$  is the Riemannian volume of the parallelepiped generated by the  $v_j$  (this is the set  $\{\sum_{j=1}^n t_j v_j : t_j \in [0, 1]\}$ ).

The volume should have the following properties if the  $v_j$  have very small (infinitesimal) length:

(a): If  $v_1, \dots, v_n$  are orthogonal with respect to  $g$ , one should have

$$\text{Vol}_g(v_1, \dots, v_n) \approx |v_1|_g \cdots |v_n|_g$$

(b): If  $A$  is a matrix with  $Av_j = \lambda_j v_j$ ,  $j = 1, \dots, n$ , one should have

$$\text{Vol}_g(Av_1, \dots, Av_n) \approx \lambda_1 \cdots \lambda_n \text{Vol}_g(v_1, \dots, v_n)$$

(c): More generally if  $A$  is any  $n \times n$  matrix, then one should have

$$\text{Vol}_g(Av_1, \dots, Av_n) \approx \det(A) \text{Vol}_g(v_1, \dots, v_n)$$

Fix now a point  $x \in U$ , write  $G = (g_{jk}(x))_{j,k=1}^n$ , and note that the set  $\{G^{-1/2}e_1, \dots, G^{-1/2}e_n\}$  is an  $g$ -orthonormal basis of  $T_x U$ :

$$\begin{aligned} \langle G^{-1/2}e_j, G^{-1/2}e_k \rangle_g &= g_{pq}(x) (G^{-1/2}e_j)^p (G^{-1/2}e_k)^q = G(G^{-1/2}e_j) \cdot (G^{-1/2}e_k) \\ &= G^{-1/2} G G^{-1/2} e_j \cdot e_k = e_j \cdot e_k = \delta_{jk}. \end{aligned}$$

Thus the volume of an infinitesimal parallelepiped should be

$$\begin{aligned} \text{Vol}_g(h e_1|_x, \dots, h e_n|_x) &\approx h^n \text{Vol}_g(G^{1/2}(G^{-1/2}e_1)|_x, \dots, G^{1/2}(G^{-1/2}e_n)|_x) \\ &\approx h^n |g(x)|^{1/2}, \end{aligned}$$

where  $|g(x)| = \det(g_{jk}(x))$ . Going back to (2.8), this would give

$$\int_U f(x) d\text{Vol}_g(x) \approx \sum_{j=1}^N f(x_j) |g(x_j)|^{1/2} h^n \xrightarrow{h \rightarrow 0} \int_U f(x) |g(x)|^{1/2} dx.$$

The above discussion motivates the following definitions:

**Definition 2.33** (Riemannian volume and integration). Let  $U \subset \mathbb{R}^n$  be open, and let  $g$  be a Riemannian metric on  $U$ . If  $f \in C_c(U)$ , we define the integral of  $f$  on  $U$  by

$$\int_U f(x) d\text{Vol}_g(x) := \int_U f(x) |g(x)|^{1/2} dx.$$

The *Riemannian volume* of a measurable set  $E \subset U$  is

$$\text{Vol}_g(E) := \int_E |g(x)|^{1/2} dx.$$

If  $1 \leq p < \infty$ , the  $L^p$  norm of  $f$  is

$$\|f\|_{L^p(U, dV_g)} := \left( \int_U |f|^p dV_g \right)^{1/p},$$

where for notational simplicity, we write  $V_g$  for  $\text{Vol}_g$ . The space  $L^p(U, dV_g)$  is the completion of  $C_c(U)$  in the  $L^p$  norm. It is easy to show that  $L^p(U, dV_g)$  is a Banach space whenever  $1 \leq p < \infty$ .

**Remark 2.34.** The quantity  $dV_g$  is usually called the *volume form* of the Riemannian manifold  $(U, g)$ . To justify this terminology, one should interpret  $dV_g$  as the differential  $n$ -form (element of  $\Omega^n(U)$ ) given by

$$dV_g = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^n.$$

One can equivalently think of  $dV_g$  as a measure, i.e. (using the Riesz representation theorem for measures) as a linear operator acting on functions in  $C_c(U)$  by

$$f \mapsto \int_U f dV_g.$$

In the present setting where  $U \subset \mathbb{R}^n$ , this measure is absolutely continuous with respect to Lebesgue measure ( $dV_g(x) = |g(x)|^{1/2} dx$ ).

**Inner products on  $L^2$ .** The most important case of  $L^p$  spaces during this course is  $p = 2$ . In fact,  $L^2(U, dV_g)$  is a Hilbert space with the following inner product.

**Definition 2.35.** If  $u, v \in L^2(U, dV_g)$ , we define

$$(u, v)_{L^2} := \int_U uv dV_g.$$

We now wish to define an  $L^2$  inner product for vector fields and tensor fields on  $U$  as well. The case of vector fields comes naturally: if  $F, G \in C_c(U, \mathbb{R}^n)$  are two vector fields, so that  $F(x), G(x) \in T_x U$  for each  $x \in U$ , the  $g$ -inner product of  $F(x)$  and  $G(x)$  is

$$(2.9) \quad \langle F(x), G(x) \rangle_g = g_{jk}(x) F^j(x) G^k(x).$$

The  $L^2$ -inner product of  $F$  and  $G$  is then defined by

$$\begin{aligned} (F, G)_{L^2} &:= \int_U \langle F(x), G(x) \rangle_g dV_g(x) \\ &= \int_U g_{jk}(x) F^j(x) G^k(x) |g(x)|^{1/2} dx. \end{aligned}$$

Next consider the case of 1-forms. Let  $\alpha$  and  $\beta$  be two 1-forms in  $U$  whose coordinate functions are in  $C_c(U)$ , meaning that  $\alpha = \alpha_j dx^j$  and  $\beta = \beta_k dx^k$ , where  $\alpha_k, \beta_k \in C_c(U)$ . If  $\alpha(x)$  denotes the expression  $\alpha_j(x) dx^j$ , in analogy with (2.9) it seems natural to define the  $g$ -inner product

$$(2.10) \quad \begin{aligned} (\alpha, \beta)_{L^2} &:= \int_U \langle \alpha(x), \beta(x) \rangle_g dV_g(x) \\ &= \int_U g^{jk}(x) \alpha_j(x) \beta_k(x) |g(x)|^{1/2} dx. \end{aligned}$$

Motivated by (2.10), one can define the  $L^2$  inner product of two tensor fields with components in  $C_c(U)$ . In particular, this gives an  $L^2$  inner product on differential forms since  $k$ -forms can be identified with certain (alternating)  $k$ -tensor fields by Theorem 2.20.

**Definition 2.36.** Let  $u = (u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$  and  $v = (v_{k_1 \dots k_m})_{k_1, \dots, k_m=1}^n$  be two tensor fields such that each  $u_{j_1 \dots j_m}$  and  $v_{k_1 \dots k_m}$  is in  $C_c(U)$ . The  $L^2$  inner product of  $u$  and  $v$  is

$$(u, v)_{L^2} := \int_U g^{j_1 k_1}(x) \cdots g^{j_m k_m}(x) u_{j_1 \dots j_m} v_{k_1 \dots k_m} |g(x)|^{1/2} dx.$$

If  $\alpha$  and  $\beta$  are differential  $k$ -forms whose component functions are in  $C_c(U)$ , we denote by

$$(\alpha, \beta)_{L^2} := (\tilde{\alpha}, \tilde{\beta})_{L^2}$$

the inner product of the corresponding tensor fields as in Theorem 2.20.

Recall that if  $\alpha = \alpha_I dx^I$  is a  $k$ -form, Theorem 2.20 identifies  $\alpha$  with the  $k$ -tensor  $\tilde{\alpha}$  defined by

$$\tilde{\alpha}_{j_1 \dots j_k} := \begin{cases} 0, & (j_1, \dots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}} \varepsilon_{j_1 \dots j_k} \alpha_{R(j_1, \dots, j_k)}, & (j_1, \dots, j_k) \text{ contains no repeated index,} \end{cases}$$

where  $R(j_1, \dots, j_k) = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$  and  $\sigma$  is the unique permutation of  $\{1, \dots, k\}$  such that  $j_1 < j_2 < \dots < j_k$  (thus  $R$  puts the indices in increasing order) and  $\varepsilon_{j_1, \dots, j_k} = (-1)^{\text{Sign}(\sigma)}$ .

Notice that if  $\alpha$  and  $\beta$  are 1-forms, this inner product is equal to (2.10).

**Example 2.37.** Let  $U \subset \mathbb{R}^n$  be open and let  $g$  be the Euclidean metric, so  $g_{jk} = \delta_{jk}$ . Then  $|g(x)| = 1$  and  $g^{jk} = \delta^{jk}$ . If  $\alpha = \alpha_j dx^j$  and  $\beta = \beta_k dx^k$  are two 1-forms with  $\alpha_j, \beta_k \in C_c(U)$ , and if  $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$  and  $\vec{\beta} = (\beta_1, \dots, \beta_n)$  are the corresponding vector fields, then

$$(\alpha, \beta)_{L^2} = \int_U \sum_{j=1}^n \alpha_j \beta_j dx = \int_U \vec{\alpha} \cdot \vec{\beta} dx.$$

Moreover, if  $u = (u_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$  and  $v = (v_{k_1 \dots k_m})_{k_1, \dots, k_m=1}^n$  are two vector fields with components in  $C_c(U)$ , then

$$(u, v)_{L^2} = \int_U \sum_{j_1, \dots, j_m=1}^n u_{j_1 \dots j_m} v_{j_1 \dots j_m} dx.$$

**Codifferential.** Our next purpose is to consider the exterior derivative  $d: \Omega^k(U) \rightarrow \Omega^{k+1}(U)$  and to compute its formal adjoint operator in the  $L^2$  inner product on forms. Below, we write  $\Omega_c^k(U)$  for the set of compactly supported  $k$ -forms in  $U$  (thus  $\alpha = \alpha_I dx^I$  is in  $\Omega_c^k(U)$  if  $\alpha_I \in C_c^\infty(U)$  for each  $I$ ).

**Theorem 2.38** (Codifferential). Let  $U \subset \mathbb{R}^n$  be open and let  $g$  be a Riemannian metric on  $U$ . For each  $k$  with  $0 \leq k \leq n$ , there is a unique linear operator

$$\delta: \Omega^k(U) \rightarrow \Omega^{k-1}(U)$$

having the property

$$(2.11) \quad (d\alpha, \beta)_{L^2} = (\alpha, \delta\beta)_{L^2}, \quad \alpha \in \Omega_c^{k-1}(U), \beta \in \Omega^k(U).$$

The operator  $\delta$  satisfies  $\delta \circ \delta = 0$  and  $\delta|_{\Omega^0(U)} = 0$ . It is a linear first order differential operator acting on component functions, and on 1-forms it is given by

$$(2.12) \quad \delta\beta := -|g|^{-1/2}\partial_j(|g|^{1/2}g^{jk}\beta_k), \beta = \beta_k dx^k \in \Omega^1(U).$$

The proof is based on the integration by parts formula

$$(2.13) \quad \int_U u(\partial_j v) dx = - \int_U (\partial u)v dx, \quad u \in C(U), v \in C_c(U).$$

*Proof.* We begin with the case  $k = 1$ . Let  $\beta = \beta dx \in \Omega^k(U)$ . To compute  $\delta\beta$  satisfying (2.11), we take  $\alpha \in \Omega_c^0(U) = C_c^\infty(U)$  and compute

$$\begin{aligned} \langle d\alpha, \beta \rangle &= \int_U \langle d\alpha, \beta \rangle dV_g = \int_U g^{jk} \partial_j \alpha \beta_k |g|^{1/2} dx \\ &= - \int_U \alpha |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \beta_k) dV_g. \end{aligned}$$

Thus (2.11) will be satisfied for  $k = 1$  if we define  $\delta: \Omega^1(U) \rightarrow \Omega^0(U)$  by (2.12).

Let us now show that for any  $k$ , there is an operator  $\delta: \Omega^k(U) \rightarrow \Omega^{k-1}(U)$  such that (2.11) holds. Let  $\alpha \in \Omega_c^{k-1}(U)$  and  $\beta \in \Omega^k(U)$ . Using the definitions and integration by parts, we obtain

$$\begin{aligned} \langle d\alpha, \beta \rangle &= \int_U \langle \partial_i \alpha_I dx^i \wedge dx^I, \beta_J dx^J \rangle_g dV_g \\ &= \int_U (\partial_i \alpha_I) \beta_J \langle dx^i \wedge dx^I, dx^J \rangle_g |g|^{1/2} dx \\ &= - \int_U \alpha_I |g|^{-1/2} \partial_i \left[ |g|^{1/2} \langle dx^i \wedge dx^I, dx^J \rangle_g \beta_J \right] dV_g. \end{aligned}$$

Write  $\gamma^I := -|g|^{-\frac{1}{2}} \partial_i \left[ |g|^{1/2} \langle dx^i \wedge dx^I, dx^J \rangle_g \beta_J \right]$ . It follows that

$$\langle d\alpha, \beta \rangle_{L^2} = \int_U \alpha_I \gamma^I dV_g.$$

We wish to find  $\gamma = \gamma_L dx^L \in \Omega^{k-1}(U)$  such that  $\alpha_I \gamma^I = \langle \alpha, \gamma \rangle_g$ . This can be done by *lowering indices*. First let  $\tilde{\alpha} = (\tilde{\alpha}_{i_1 \dots i_{k-1}})$  and  $\tilde{\gamma} = (\tilde{\gamma}^{i_1 \dots i_{k-1}})$  be the alternating tensor fields corresponding to  $\alpha_I$  and  $\gamma^I$ , so for instance  $\tilde{\gamma}^{i_1 \dots i_{k-1}} := \frac{1}{\sqrt{(k-1)!}} \varepsilon^{i_1 \dots i_{k-1}} \gamma^{R(i_1, \dots, i_{k-1})}$ .

Let

$$\tilde{\gamma}_{l_1 \dots l_{k-1}} := g_{l_1 i_1} \cdots g_{l_{k-1} i_{k-1}} \tilde{\gamma}^{i_1 \dots i_{k-1}}$$

and let  $\gamma = \gamma_L dx^L$  be the  $(k-1)$ -form corresponding to  $\tilde{\gamma}$ . Then

$$\begin{aligned} \langle \alpha, \gamma \rangle_g &= \langle \tilde{\alpha}, \tilde{\gamma} \rangle_g = g^{i_1 l_1} \cdots g^{i_{k-1} l_{k-1}} \tilde{\alpha}_{i_1 \dots i_{k-1}} [g_{l_1 p_1} \cdots g_{l_{k-1} p_{k-1}} \tilde{\gamma}^{p_1 \dots p_{k-1}}] \\ &= \tilde{\alpha}_{i_1 \dots i_{k-1}} \tilde{\gamma}^{i_1 \dots i_{k-1}} = \frac{1}{(k-1)!} \alpha_{R(i_1 \dots i_{k-1})} \gamma^{R(i_1 \dots i_{k-1})} = \alpha_I \gamma^I. \end{aligned}$$

Combining the above arguments, we have proved that

$$(d\alpha, \beta)_{L^2} = (\alpha, \gamma)_{L^2}$$

for all  $\alpha \in \Omega_c^{k-1}(U)$ . Here  $\gamma \in \Omega^{k-1}(U)$  is determined uniquely by this identity, thus setting  $\delta\beta := \gamma$  satisfies (2.12). Insepecting the above argument shows that  $\delta\beta = \gamma_L dx^L$ , where for  $L = (l_1, \dots, l_{k-1})$ ,

$$\gamma_L = -g_{l_1 i_1} \cdots g_{l_{k-1} i_{k-1}} |g|^{-\frac{1}{2}} \partial_i [|g|^{1/2} \langle dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}, dx^J \rangle_g \beta_J].$$

Thus  $\delta$  is a first order operator acting on the component functions  $\beta_J$ .

It is clear that  $\delta|_{\Omega^0(U)} = 0$ , and the condition  $\delta \circ \delta = 0$  follows from (2.11) and the fact that  $d \circ d = 0$ .  $\square$

If  $U \subset \mathbb{R}^n$  is an open set, in Section 2.3, we studied the sequence

$$(2.14) \quad \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U),$$

where  $d \circ d = 0$ . This sequence does not depend on any Riemannian metric on  $U$ . However, if we introduce a Riemannian metric  $g$  on  $U$ , then Theorem 2.38 shows that there is another sequence

$$(2.15) \quad \Omega^0(U) \xleftarrow{\delta} \Omega^1(U) \xleftarrow{\delta} \cdots \xleftarrow{\delta} \Omega^{n-1}(U) \xleftarrow{\delta} \Omega^n(U),$$

where  $\delta \circ \delta = 0$ . As we will explain later, the sequences (2.14) and (2.15) and the corresponding cohomology groups turn out to be dual to each other: this is related to *Poincaré duality*.

**2.7. Laplace-Beltrami operator.** In this section we will see that on any open set equipped with a Riemannian metric, there is a canonical second order elliptic operator, called the *Laplace-Beltrami operator*, which is an analogue of the usual Laplacian in  $\mathbb{R}^n$ .

**Motivation.** Let first  $U$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary, and consider the Laplace operator

$$(2.16) \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

Solutions of the equation  $\Delta u = 0$  are called *harmonic functions*, and by standard results for elliptic PDE, for any  $f \in H^1(U)$ , there is a unique solution  $u \in H^1(U)$  of the Dirichlet problem

$$(2.17) \quad \begin{cases} -\Delta u = 0 & \text{in } U, \\ u = f & \text{on } \partial U. \end{cases}$$

The last line means that  $u - f \in H_0^1(U)$ .

One way to produce the solution of (2.17) is based on variational methods and Dirichlet's principle (see [5]). We define the Dirichlet energy

$$E(v) := \frac{1}{2} \int_U |\nabla v|^2 dx, \quad v \in H^1(U).$$

If we define the admissible class

$$\mathcal{A}_f := \{v \in H^1(U) : v = f \text{ on } \partial U\},$$



then the solution of (2.17) is the unique function  $u \in \mathcal{A}_f$  which minimizes the Dirichlet energy:

$$E(u) \leq E(v) \quad \text{for all } v \in \mathcal{A}_f.$$

The heuristic idea is that the solution of (2.17) represents a physical system in equilibrium, and therefore should minimize a suitable energy functional. The point is that one can start from the energy functional  $E(\cdot)$  and conclude that any minimizer  $u$  must satisfy  $\Delta u = 0$ , which gives another way to define the Laplace operator.

From this point on, let  $U \subset \mathbb{R}^n$  be open and let  $g$  be a Riemannian metric on  $U$ . Although there is no immediately obvious analogue of (2.16) that would take into account the metric  $g$ , there is a natural analogue of the Dirichlet energy. It is given by

$$E(v) := \frac{1}{2} \int_U |dv|^2 dV, \quad v \in H^1(U).$$

Here  $|dv|$  is the Riemannian length of the 1-form  $dv$ , and  $dV$  is the volume form.

We wish to find a differential equation which is satisfied by minimizers of  $E(\cdot)$ . Suppose  $u \in H^1(U)$  is a minimizer which satisfies  $E(u) \leq E(u + t\varphi)$  for all  $t \in \mathbb{R}$  and all  $\varphi \in C_c^\infty(U)$ . We have

$$\begin{aligned} E(u + t\varphi) &= \frac{1}{2} \int_U \langle d(u + t\varphi), d(u + t\varphi) \rangle dV \\ &= E(u) + t \int_U \langle du, d\varphi \rangle dV + t^2 E(\varphi). \end{aligned}$$

Since  $I_\varphi(t) := E(u + t\varphi)$  is a smooth function of  $t$  for fixed  $\varphi$ , and since  $I_\varphi(0) \leq I_\varphi(t)$  for  $|t|$ -small, we must have  $I'_\varphi(0) = 0$ . This shows that if  $u$  is a minimizer, then

$$\int_U \langle du, d\varphi \rangle dV = 0$$

for any choice of  $\varphi \in C_c^\infty(U)$ . By the properties of the codifferential  $\delta$ , this implies that

$$\int_U (\delta du) \varphi dV = 0$$

for all  $\varphi \in C_c^\infty(U)$ . Thus any minimizer  $u$  has to satisfy the equation

$$\delta du = 0 \quad \text{in } U.$$

We have arrived at the definition of the Laplace-Beltrami operator.

**Definition 2.39** (Laplace-Beltrami operator). The Laplace-Beltrami operator on  $(U, g)$  is defined by

$$\Delta_g u := -\delta du.$$

**Lemma 2.40.** The Laplace-Beltrami operator has the expression

$$\Delta_g u = |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k u),$$

where, as before,  $|g| = \det(g_{jk})$  is the determinant of  $g$ .

*Proof.* Exercise. □

**Remark 2.41.** There are differing sign conventions for the Laplace-Beltrami operator. Honoring the title of this course, we have chosen the convention which is perhaps most common in analysis and makes the Laplace-Beltrami operator for Euclidean metric equal to  $\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ . However, it is very common in geometry define the Laplace-Beltrami operator with the opposite sign, which has the benefit that the operator becomes positive. Moreover, in probability theory a factor of  $\frac{1}{2}$  is often included in the definition. In this course we will stick to the analysts' convention so that  $\Delta_g = -\delta d$ .

The existence of a canonical Laplace operator associated to a Riemannian metric implies that one has analogues of the classical linear PDE:

- $\Delta_g u = 0$  (Laplace)
- $\partial_t u - \Delta_g u = 0$  (heat)
- $\partial_t^2 u - \Delta_g u = 0$  (wave)
- $i\partial_t u + \Delta_g u = 0$  (Schrödinger)

Therefore in physical terms, any Riemannian manifold will support a theory for electrostatics, heat flow, acoustic wave propagation, and quantum mechanics. Note also that the theory of geodesics leads to a version of classical mechanics, and there are many relations between the classical and quantum picture (i.e. between the geodesic flow and the Laplace-Beltrami operator).

#### REFERENCES

- [1] M. BERGER, *A panoramic view of Riemannian geometry*. Springer 2002.
- [2] I. CHAVEL, *Riemannian geometry. A modern introduction*. 2nd edition, Cambridge University Press, 2006.
- [3] S.-S. CHERN, *Lectures on differential geometry*, World Scientific Publishing Co., 1999.
- [4] M.P. DO CARMO, *Riemannian geometry*. 1992.
- [5] L.C. EVANS, *Partial differential equations*. 2nd edition, AMS, 2010.
- [6] J.M. LEE, *Riemannian manifolds. An introduction to curvature*. Springer, 1997.
- [7] J.M. LEE, *Introduction to smooth manifolds*. Springer, 2002.
- [8] I. MADSEN AND J. TORNEHAVE, *From calculus to cohomology*. Cambridge University Press, 1997.
- [9] P. PETERSEN, *Riemannian geometry*. 2nd edition, Springer, 2006.
- [10] M.E. TAYLOR, *Partial differential equations I. Basic theory*. Springer 1996.