Exercise Four

1. Let $\langle \cdot, \cdot \rangle$ be an iner product on $\mathbb{R}^n$. Show that there is some positive definite symmetric matrix $A = (a_{jk})_{j,k=1}^n$ such that

$$\langle v, w \rangle = Av \cdot w, \quad v, w \in \mathbb{R}^n.$$

2. Let $\gamma\colon [0,1] \to \mathbb{R}^n$ be a regular curve. Show that $L(\gamma)$ is independent of the way γ is parametrized.

3. Let $\gamma\colon [0,1] \to \mathbb{R}^n$ be a regular curve. Show that γ can be parametrized by arc-length, i.e. $|\dot{\gamma}(t)| \equiv 1$ for all t.

4. Let $U \subset \mathbb{R}^n$ be a connected bounded open set and let g be a Riemannian metric on U. Show that for each pair of points $x, y \in U$, there exists a curve γ such that γ minimizes the distance, i.e. for any other curve β that connects x and y in U, we must have $L(\gamma) \le L(\beta)$.

5. Let $U \subset \mathbb{R}^n$ be an open connected set and let g be a Riemannian metric on U. We denote by g_0 the standard Euclidean distance on U. Show that

- For each compact subset $K \subset U$, there exists positive constants c and C such that for all $v \in T_x\mathbb{R}^n$,

$$c|v|_{g_0} \le |v|_g \le C|v|_{g_0}.$$

- Show that with the Riemannian distance d_g, $U = (U, d_g)$ becomes a metric space whose topology is the same as the Euclidean topology on U.