Correction Serie 4/5

April 1, 2019

Exercise 2. Soit $\Gamma = \mathbb{Z}u + \mathbb{Z}v$ un reseau avec une base donnee.

1. In order to show that the set $\mathrm{End}_{\mathcal{L}^2}(\Gamma)$ of endomorphisms of the lattice can be identified with $M_2(\mathbb{Z})$, we define the following bijective map

$$\Psi : \mathrm{End}_{\mathcal{L}^2}(\Gamma) \to M_2(\mathbb{Z})$$

as follows. Let $\phi \in \mathrm{End}_{\mathcal{L}^2}(\Gamma)$. The action of ϕ is defined by its action on the two basis vectors of the lattice, u and v. Let $\phi(u) = au + cv$ and $\phi(b) = bu + dv$. Then define

$$\Psi(\phi) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This map is clearly injective, by construction. If $\Psi(\phi) = \Psi(\varphi)$ for two lattice endomorphisms, then ϕ and φ coincide on the basis vectors, hence $\phi = \varphi$. In order to show surjectivity, let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$. The endomorphism ϕ defined on the basis vectors by $\phi(u) = au + cv$ and $\phi(b) = bu + dv$ satisfies $\Psi(\phi) = M$ by construction. Hence Ψ is surjective.

2. We already know that $\mathrm{GL}_2(\mathbb{R})$ is a group. So in order to show that $\mathrm{GL}_2(\mathbb{Z})$ is a group, we can show that it is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{Z}$. Its inverse is $M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -a & a \end{bmatrix}$, and hence since $ad - bc = \pm 1$, $M^{-1} \in \mathrm{GL}_2(\mathbb{Z})$. In order to show that $\mathrm{GL}_2(\mathbb{Z})$ is closed under multiplication, take $M, N \in \mathrm{GL}_2(\mathbb{Z})$. It holds that $M * N \in \mathrm{GL}_2(\mathbb{Z})$, since clearly the entries are integers, and furthermore $\det(M * N) = \det M \det N = \pm 1$.

3. We want to show that

$$\Psi(\mathrm{Aut}_{\mathcal{L}^2}(\Gamma)) = \mathrm{GL}_2(\mathbb{Z})$$

holds. First, let $\phi \in \mathrm{Aut}_{\mathcal{L}^2}(\Gamma)$. Since ϕ is bijective, there exists $\varphi \in \mathrm{Aut}_{\mathcal{L}^2}(\Gamma)$ such that $\phi \circ \varphi = \varphi \circ \phi = \mathrm{Id}$. But since $\Psi(\phi) * \Psi(\varphi) = \Psi(\phi \circ \varphi) = \mathrm{Id}_2$, the 2×2 identitiy matrix, we get, passing to the determinant,

$$\begin{aligned} &\det(\Psi(\phi) * \Psi(\varphi)) = \det(\mathrm{Id}_2) \\ \Leftrightarrow\ &\det(\Psi(\phi) * \Psi(\varphi)) = 1 \end{aligned}$$

Since the determinants are integers, it follws that $\det(\Psi(\phi)) = \pm 1$, and hence $\Psi(\phi) \in \mathrm{GL}_2(\mathbb{Z})$.

In order to show the other direction, let $M \in \mathrm{GL}_2(\mathbb{Z})$. We want to show that $\Psi^{-1}(M) \in \mathrm{Aut}_{\mathcal{L}^2}(\Gamma)$, i.e. that $\Psi^{-1}(M)$ is bijective. Since $M \in \mathrm{GL}_2(\mathbb{Z})$, there exists an inverse matrix $M^{-1} \in \mathrm{GL}_2(\mathbb{Z})$. It holds that $\Psi^{-1}(M) \circ \Psi^{-1}(M^{-1}) = \Psi^{-1}(M * M^{-1}) = \Psi^{-1}(\mathrm{Id}_2) = \mathrm{Id}$, which shows that $\Psi^{-1}(M)$ is bijective.

4. Let u', v' be a basis of Γ. We can write $u' = au + cv$, $v' = bu + dv$. Let $\phi \in \mathrm{End}_{\mathcal{L}^2}(\Gamma)$ be defined by $\phi(u) = u'$ and $\phi(v) = v'$. This is a bijective morphism, and therefore $\phi \in \mathrm{Aut}_{\mathcal{L}^2}(\Gamma)$. By 3. it follows that the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, and so $ad - bc = \pm 1$.

   On the other hand, if we have

   $$u' = au + cv, \quad v' = bu + dv,$$

   with $ad - bc = \pm 1$, then by 3., this corresponds to an automorphism ϕ defined by $u' = \phi(u) = au + cv$, $v' = \phi(v) = bu + dv$. It follows that (u', v') is a basis of Γ.

5. Let $\mathbf{P}_{u,v}$ be the parallelogram spanned by the two vectors u, v. The surface area is equal to the length of the cross product $u \times v$,

   $$\mathrm{Aire}(\mathbf{P}_{u,v}) = \left|u \times v\right| = \left|u_x v_y - u_y v_x\right| = \Bigg| \det \underbrace{\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}}_{:=M_{u,v}} \Bigg|,$$

   where $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$, and equivalently for v.

   Let (u', v') be an other basis of the lattice Γ. Let $M_{u,v}$ respecitvely $M_{u',v'}$ be the matrices corresponding to (u, v), repsectivley (u', v'). Denote by $B \in \mathrm{GL}_2(\mathbb{Z})$ the matrix of the base change, as defined in 4., with $M_{u,v} * B = M_{u',v'}$. It follows that

   $$\mathrm{Aire}(\mathbf{P}_{u',v'}) = \left|\det M_{u',v'}\right| = \left|\det(M_{u,v} * B)\right| = \left|\det M_{u,v}\right|\left|\det B\right| = \left|\det M_{u,v}\right| = \mathrm{Aire}(\mathbf{P}_{u,v})$$

Exercise 3. 1. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $L = \mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}\begin{bmatrix} u_x \\ u_y \end{bmatrix} + \mathbb{Z}\begin{bmatrix} v_x \\ v_y \end{bmatrix} \in \mathcal{L}_2$.

We define the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{L}_2$ as follows

$$M.L = \mathbb{Z}M.u + \mathbb{Z}M.v = \mathbb{Z}\begin{bmatrix} au_x + bu_y \\ cu_x + du_y \end{bmatrix} + \mathbb{Z}\begin{bmatrix} av_x + bv_y \\ cv_x + dv_y \end{bmatrix}$$

This is clearly an action, as one can easily check.

2. In order to show that this action is transitive, let $\Gamma = \mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}\begin{bmatrix} u_x \\ u_y \end{bmatrix} + \mathbb{Z}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$. Let e_1, e_2 be the two standard basis vectors of $\mathbb{R}^2$. Then it holds that with $M = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$

$$M.(\mathbb{Z}e_1 + \mathbb{Z}e_2) = \mathbb{Z}M.e_1 + \mathbb{Z}M.e_2 = \mathbb{Z}\begin{bmatrix} u_x \\ u_y \end{bmatrix} + \mathbb{Z}\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \mathbb{Z}u + \mathbb{Z}v.$$

This means that there exists only one orbit, and so the action is transitive.

3. The stabiliser of the lattice $\mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2$ is the set of matrices M for which $M.\mathbb{Z}^2 = \mathbb{Z}^2$ holds. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with $M.\mathbb{Z}^2 = \mathbb{Z}^2$. It holds that

$$M.\mathbb{Z}^2 = M.(\mathbb{Z}e_1 + \mathbb{Z}e_2) = \mathbb{Z}M.e_1 + \mathbb{Z}M.e_2 = \mathbb{Z}\underbrace{\begin{bmatrix} a \\ c \end{bmatrix}}_{=:u} + \mathbb{Z}\underbrace{\begin{bmatrix} b \\ d \end{bmatrix}}_{=:v}.$$

If M is contained in the stabiliser, then the vectors $u = ae_1 + ce_2$ and $v = be_1 + de_2$ form a basis of $\mathbb{Z}^2$. By question 4. of the previous exercise, this holds exactly if $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \pm 1$, which by definition means that $M \in \mathrm{GL}_2(\mathbb{Z})$. By the Orbit/Stabiliser Theorem, there is an isomorphism

$$\mathcal{L}_2 \simeq \mathrm{GL}_2(\mathbb{R}) / \mathrm{GL}_2(\mathbb{Z}).$$

4. Let $M \in \mathrm{GL}_2(\mathbb{R})$ and Γ a lattice. Then

$$\mathrm{vol}(M \cdot \Gamma) = \left|\det M\right| \mathrm{vol}(\Gamma)$$

is a direct consequence of the definiton of the volume in question 5. of the previous exercise.

Exercise 4. For $z \in \mathbb{C}^*$ non-real, we define the lattice

$$\Gamma_z := \mathbb{Z} + \mathbb{Z}.z$$

with basis $(1, z)$.

1. The volume of Γ_z is equal to the determinant of the matrix M that contains in its columns the coordinates of 1 and z in the standard basis. Let $z = x + iy$. Then $M = \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix}$. Hence $\mathrm{vol}(\Gamma_z) = \left|\det M\right| = |y|$.

2. As seen in Serie 3,

$$\mathcal{D}_{\mathrm{SL}_2(\mathbb{Z})} = \{z \in \mathbb{H} |\, \mathrm{Re}(z) \in [-1/2, 1/2[, |z| > 1\} \cup \{z \in \mathbb{H} |\, \mathrm{Re}(z) \in [-1/2, 0], |z| = 1\}$$

is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Let $z = x + iy \in \mathcal{D}_{\mathrm{SL}_2(\mathbb{Z})}$, and define the lattice $\Gamma_z = \mathbb{Z}.1 + \mathbb{Z}.z$.

First we show that 1 is the element of smallest length in Γ_z. Let $\gamma = c + dz \in \Gamma_z$, with $c, d \in \mathbb{Z}$, not both equal to zero. We want to show that $|\gamma| \geq 1$ holds.

- If $d = 0$, then $\gamma = c$ and so $|\gamma| = |c| \geq 1$.
- If $d = 1$, then $\gamma = c + z$ and

$$\begin{aligned}|\gamma| &= |c + (x + iy)| = \sqrt{(c + x)^2 + y^2} = \sqrt{c^2 + 2cx + x^2 + y^2} \\ &\overset{*}{\geq} \sqrt{c^2 + 2cx + 1} \overset{**}{\geq} \sqrt{-c + c^2 + 1} \geq 1.\end{aligned}$$

  The inequality $*$ follows from the fact that $x^2 + y^2 = |z|^2 \geq 1$ in $\mathcal{D}_{\mathrm{SL}_2(\mathbb{Z})}$ and at $**$ we use the fact that $x \geq -1/2$, and so $2cx \geq -c$.
- If $d = -1$, a similar computation to the one above shows that $|\gamma| \geq 1$.
- If $|d| \geq 2$, we consider

$$\begin{aligned}|\gamma| = |c + dz| &= \sqrt{(\mathrm{Re}(c + dz))^2 + (\mathrm{Im}(c + dz))^2} \\ &\geq |\,\mathrm{Im}(c + dz)| = |d||\,\mathrm{Im}(z)| \geq |d|\frac{\sqrt{3}}{2} > 1.\end{aligned}$$

  where the last inequality holds due to the fact that $|z| \geq 1$, and so

$$\sqrt{x^2 + y^2} \geq 1 \Rightarrow y^2 \geq 1 - x^2 \geq 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} \Rightarrow y \geq \frac{\sqrt{3}}{2}.$$

Next we show that z is minimal amongst the elements in the lattice that are non-collinear to 1. Let $\gamma = c + dz \in \Gamma_z$, with $c, d \in \mathbb{Z}, d \neq 0$. We want to show that $|\gamma| \geq |z|$ holds.

- For $d = 1$, we have $\gamma = c + z$, and so

$$\begin{aligned}|\gamma| &= |c + x + iy| = \sqrt{(c + x)^2 + y^2} = \sqrt{c^2 + 2cx + x^2 + y^2} \\ &\overset{*}{\geq} \sqrt{c^2 - c + x^2 + y^2} \geq \sqrt{x^2 + y^2} = |z|.\end{aligned}$$

  The inequality * holds because $x \geq -1/2$.
- Similarly for $d = -1$.
- Let $\gamma = c + dz$ with $|d| \geq 2$. We want to show that

$$\begin{aligned}&|\gamma| \geq |z| \\ \Leftrightarrow\ &|\gamma|^2 \geq |z|^2 \\ \Leftrightarrow\ &|c + d(x + iy)|^2 \geq |x + iy|^2 \\ \Leftrightarrow\ &\underbrace{(c + dx)^2}_{\geq 0} + (dy)^2 - x^2 - y^2 \geq 0\end{aligned}$$

  We show that

$$\begin{aligned}&d^2y^2 - x^2 - y^2 \geq 0 \\ \Leftrightarrow\ &(d^2 - 1)y^2 - x^2 \geq 0.\end{aligned}$$

Since $|z| \geq 1 \Rightarrow y^2 \geq 1 - x^2$, and so

$$(d^2-1)y^2-x^2 \geq (d^2-1)(1-x^2)-x^2 = d^2(1-x^2)-1+x^2-x^2 = d^2\underbrace{(1-x^2)}_{\geq 3/4}-1 \geq 0,$$

since $|d| \geq 2$.

Exercise 6. 1. Let $\mathbf{P}' = \gamma + \mathbf{P}$ be the translation of the fundamental parallelogram $\mathbf{P}$ by $\gamma \in \Gamma$. Suppose that $\mathbf{P}'$ intersects the circle $C(0, R)$. The distance between any two points contained in $\mathbf{P}'$ is at most $2r_0$. Hence if $\mathbf{P}'$ intersects $C(0, R)$, then the distance between any point of $\mathbf{P}'$ and the origin 0 is at most $R + 2r_0$, and so $\mathbf{P}' \subset B(0, R + 2r_0)$. Similarly, no point of $\mathbf{P}'$ intersects the ball $B(0, R - 3r_0)$.

2. Let N denote the number of parallelograms $\gamma + \mathbf{P}$ that are contained in $C(0, R)$. By 1., it holds that the surface area of the union of all these parallelograms contianed in $C(0, R)$ is equal to Aire$(\mathbf{P})N$, which is bounded above (respectively below) by the circle $C(0, R + 2r_0)$, respectively $R(0, R - 2r_0)$. It follows that

$$\begin{aligned}
\text{surface area of } C(0, R - 3r_0) &\leq \text{Aire}(P)N \leq \text{surface area of } C(0, R + 2r_0) \\
\Leftrightarrow \pi(R - 3r_0)^2 &\leq \text{vol}(\Gamma)N \leq \pi(R + 2r_0)^2 \\
\Leftrightarrow \frac{\pi(R - 3r_0)^2}{\text{vol}(\Gamma)} &\leq N \leq \frac{\pi(R + 2r_0)^2}{\text{vol}(\Gamma)} \\
\Leftrightarrow \frac{\pi(R^2 - 6Rr_0 + 9r_0^2)}{\text{vol}(\Gamma)} &\leq N \leq \frac{\pi(R^2 + 4Rr_0 + 4r_0^2)}{\text{vol}(\Gamma)} \\
\Leftrightarrow \frac{\pi R^2}{\text{vol}(\Gamma)} + \frac{\pi(-6Rr_0 + 9r_0^2)}{\text{vol}(\Gamma)} &\leq N \leq \frac{\pi R^2}{\text{vol}(\Gamma)} + \frac{\pi(4Rr_0 + 4r_0^2)}{\text{vol}(\Gamma)} \\
\Leftrightarrow \frac{\pi(-6Rr_0 + 9r_0^2)}{\text{vol}(\Gamma)} &\leq \underbrace{N - \frac{\pi R^2}{\text{vol}(\Gamma)}}_{=:A} \leq \frac{\pi(4Rr_0 + 4r_0^2)}{\text{vol}(\Gamma)}
\end{aligned}$$

Hence $N = \frac{\pi R^2}{\text{vol}(\Gamma)} + A$, where A is a function depending on R.

3. We show that the limit $\delta(\Gamma, r) = \lim_{R\to\infty} \frac{\text{Aire}(B(0,R)\cap\Gamma(r))}{\pi R^2}$ exists for r satisfying the condition that

$$\gamma \neq \gamma' \in \Gamma \Rightarrow B(\gamma, r) \cap B(\gamma', r) = \emptyset, \tag{1}$$

and that its value is $\frac{\pi r^2}{\text{vol}(\Gamma)}$.

According to the previous question, there are $\sim \frac{\pi R^2}{\text{vol}(\Gamma)}$ lattice points that are contained in $C(0, R)$. For each of this points, we get a ball with radius r that is contained in $C(0, R)$. The surface are of all of these balls is equal to $\frac{\pi R^2}{\text{vol}(\Gamma)}\pi r^2$, and so it follows that

$$\delta(\Gamma, r) = \lim_{R\to\infty} \frac{\text{Aire}(B(0, R) \cap \Gamma(r))}{\pi R^2} \sim \frac{\frac{\pi R^2}{\text{vol}(\Gamma)}\pi r^2}{\pi R^2} \sim \frac{\pi r^2}{\text{vol}(\Gamma)}.$$

4. Let $\gamma_0, \gamma_1 \in \Gamma$ such that $\Gamma = \mathbb{Z}\gamma_0 + \mathbb{Z}\gamma_1$. Suppose that $|\gamma_0| \leq |\gamma_1|$. Then the maximal radius r_Γ, for which (1) is satisfied is $r_\Gamma = \frac{|\gamma_0|}{2}$. This can be seen by considering the translation of the fundamental parallelogram. In order for a radius r to satisfy condition (1), a ball centered in the middle of the parallelogram needs to be contained inside the parallelogram. This holds for a radius of maximal length $\frac{|\gamma_0|}{2}$, where $|\gamma_0|$ denotes the smaller side of the parallelogram.

   We define $\delta(\Gamma) = \delta(\Gamma, r_\Gamma) = \frac{\pi r_\Gamma^2}{\mathrm{vol}(\Gamma)}$.

5. We want to show that $\delta(\Gamma)$ doesn't vary under homothetie and under rotation.

   Homothetie Let the lattice $\alpha\Gamma$ be defined by the two basis vectors $\alpha\gamma_0 = \begin{bmatrix} \alpha\gamma_{0,x} \\ \alpha\gamma_{0,y} \end{bmatrix}$, where $\gamma_0 = \begin{bmatrix} \gamma_{0,x} \\ \gamma_{0,y} \end{bmatrix}$ and $\alpha\gamma_1$, for $\alpha \in \mathbb{R}$. It holds that

$$r_{\alpha\Gamma} = \frac{|\alpha\gamma_0|}{2} = \frac{|\alpha||\gamma_0|}{2} = |\alpha| r_\Gamma$$

   and

$$\mathrm{vol}(\alpha\Gamma) = \mathrm{Aire}(P_{\alpha\gamma_0,\alpha\gamma_1}) = |\alpha\gamma_0 \times \alpha\gamma_1| = \alpha^2 |\gamma_0 \times \gamma_1| = \alpha^2\, \mathrm{Aire}(P_{\gamma_0,\gamma_1}) = \alpha^2\, \mathrm{vol}(\Gamma).$$

   Therefore

$$\delta(\alpha\Gamma) = \delta(\alpha\Gamma, r_{\alpha\Gamma}) = \frac{\pi r_{\alpha\Gamma}^2}{\mathrm{vol}(\alpha\Gamma)} = \frac{\pi(|\alpha| r_\Gamma)^2}{\alpha^2\, \mathrm{vol}(\Gamma)} = \frac{\pi r_\Gamma^2}{\mathrm{vol}(\Gamma)} = \delta(\Gamma).$$

   Rotation Let $M_{rot} = \begin{bmatrix} c & -s \\ x & c \end{bmatrix} \in \mathrm{SO}_2(\mathbb{R})$ be the matrix of a rotation with angle $\theta, c = \cos\theta, s = \sin\theta$. By notation of exercise 3, the rotation of the lattice Γ is defined to be

$$\Gamma_{rot} := M_{rot}\Gamma = \mathbb{Z}M_{rot}\gamma_0 + \mathbb{Z}M_{rot}\gamma_1.$$

   It holds that

$$r_{\Gamma_{rot}} = \frac{|\gamma_0|}{2} = r_\Gamma,$$

   since the rotation does not change the length of the basis vectors. Furthermore,

$$\mathrm{vol}(\Gamma_{rot}) = \mathrm{vol}(M_{rot}\Gamma) = |\det M_{rot}|\, \mathrm{vol}(\Gamma) = \mathrm{vol}(\Gamma),$$

   since the determinant of the rotation matrix is one. Therefore

$$\delta(\Gamma_{rot}) = \delta(\Gamma_{rot}, r_{\Gamma_{rot}}) = \frac{\pi r_{\Gamma_{rot}}^2}{\mathrm{vol}(\Gamma_{rot})} = \frac{\pi r_\Gamma^2}{\mathrm{vol}(\Gamma)} = \delta(\Gamma).$$

6. We suppose that $\Gamma = \Gamma_z = \mathbb{Z} + \mathbb{Z}z$, with $z \in \mathcal{D}_{\mathrm{SL}_2(\mathbb{Z})}$. We want to show that $\delta(\Gamma_z) = \frac{\pi r_{\Gamma_z}^2}{\mathrm{vol}(\Gamma_z)}$ is maximal for $z = \omega_3 = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The value for r_{Γ_z} is defined by

the value of the first basis vector 1, and is therefore independant of the z. Hence in order for $\delta(\Gamma_z)$ to be maximal, we choose z in a way that $\mathrm{vol}(\Gamma_z)$ is minimal. Let $z = x + iy$. Then $\mathrm{vol}(\Gamma_z) = y$. Since $|z| \geq 1$ it follows that

$$\sqrt{x^2 + y^2} \geq 1 \Rightarrow y^2 \geq 1 - \underbrace{x^2}_{\in[0,(1/2)^2]} .$$

The imaginary part y is minimal if x^2 is maximal, hence if $x = \pm\frac{1}{2}$, and if the inequality $y^2 \geq 1 - x^2$ is an equality, hence if $|z| = 1$. In that case $x = -\frac{1}{2}$ and $y = \sqrt{1 - x^2} = \frac{\sqrt{3}}{2}$.