Série 6 Sol

Exercice 1. Soit $G = G_{\mathcal{P}}$ un groupe cristallographique et G^+ son sous-groupe des rotations et $T_G = T(\Gamma) \in G^+$ son reseau des translations. On note G_0 l'image de G par le morphisme partie lineaire et G_0^+ celle de G^+. On suppose que $G \neq G^+$ et on note s un element de $G - G^+$ (si il existe). On note s_0 sa partie lineaire.

1. $G^+ = ker(\det)$, therefore G^+ is a normal subgroup of G. Here det is the determinant morphism, sending elements of G to $\{\pm 1\}$.

   To show T_G is normal in G, we note that $T_G = ker(lin)$, where lin is the "linear part" morphism from G to G_0.
2. If there is no $s \in G - G^+$, then $G = G^+$ and G/G^+ is order 1. Otherwise, given any $s \in G - G^+$, we have $G = G^+ \sqcup s \cdot G^+ = G^+ \sqcup G^+ \cdot s$, and G/G^+ is of order 2.
3. We recall that $|G_0^+| = 1, 2, 3, 4, 6$. In particular, $|G^+/T_G| = |G_0^+| \leqslant 6$, since $G^+/T_G \simeq G_0^+$. Recall also $|G/G^+| \leqslant 2$. Then $|G/T_G| = |G/G^+||G^+/T_G| \leqslant 2 \cdot 6 = 12$.
4. $G = G^+ \sqcup sG^+$ and $G_0 = G_0^+ \sqcup sG_0^+$. Since $|G_0^+| = 1, 2, 3, 4, 6$, $|G_0| = 2|G_0^+| = 2, 4, 6, 8, 12$. Let $G_0^+ = \langle r \rangle$, where r is a rotation. Then $G_0 = \langle s_0, r \rangle$ is a Dihedral group.
5. We write $s = t_\delta \circ s_0$, where $\delta \in \Gamma$. Then $s^2 = t_\delta \circ s_0 \circ t_\delta \circ s_0 = t_\delta \circ t'_\delta = t_{\delta+\delta'} \in T(\Gamma)$. Here in the second equality we used the fact that T_G is normal in G.
6. We recall from the first semester (Corollary 3.1) that we can write $s = t_\gamma \circ s'$ with $s^2 = t_{2\gamma}$. From the calculation in Part 5, this implies that $2\gamma = \delta + \delta' \in \Gamma$. Hence $\gamma = \frac{\delta+\delta'}{2} \in \frac{1}{2}\Gamma$.
7. Let $t_\gamma \in T(\Gamma)$ be a translation. Then one can check that $s_0 \circ t_\gamma \circ s_0 = t_{s_0(\gamma)}$. Since $t(\Gamma)$ is a normal subgroup of G, this implies $t_{s_0(\gamma)} \in T(\Gamma)$ and hence $s_0(\gamma) \in \Gamma$. This proves the inclusion $s_0(\Gamma) \subset \Gamma$. On the other hand, for any $\gamma \in \Gamma$, we can write $\gamma = s_0(s_0(\gamma)) = s_0\gamma'$ for some $\gamma' \in \Gamma$ by what we just proved (that $s_0(\Gamma) \subset \Gamma$). This implies the other inclusion $\Gamma \subset s_0(\Gamma)$, so that we have $s_0(\Gamma) = \Gamma$.
8. We know $s_0(\Gamma) = \Gamma$. Then $s_0(\mathbb{Z}.1 + \mathbb{Z}\gamma_1) = s_0(\mathbb{Z}.1) + s_0(\mathbb{Z}\gamma_1) = \mathbb{Z}.1 + \mathbb{Z}\gamma_1$. Since s_0 is an isometry and $1 < |\gamma_1|$, we know $s_0(\mathbb{Z}.1) \neq \mathbb{Z}\gamma_1$. Then $s_0(\mathbb{Z}.1) = \mathbb{Z}.1$. By the property that 1 is of minimal length, $s_0(1) = \pm 1$.

9. Since $s_0(1) = -1$, s_0 is a symmetry whose line of symmetry is the imaginary axis. Recall $s_0(\Gamma) = \Gamma$ implies $s_0(\mathbb{Z}.1) + s_0(\mathbb{Z}\gamma_1) = \mathbb{Z}.1 + \mathbb{Z}\gamma_1$, which implies $\mathbb{Z}.1 + \mathbb{Z}\gamma_1 = \mathbb{Z}.(-1) + \mathbb{Z}.(-\overline{\gamma_1})$ and then $\mathbb{Z}.2\Re e\gamma_1 = \mathbb{Z}(\gamma_1 + \overline{\gamma_1}) = \mathbb{Z}$. Therefore $\Re e\gamma_1 \in \frac{1}{2}\mathbb{Z}$.

Exercice 2.

Exercice 3. Soient X un espace affine de dimension d et

$$P_0, \cdots, P_d \in X$$

$d+1$ points en position generale (tels que $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forment une base de V).

1. Since $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forms a basis for V, for any $P \in X$, the vector $\vec{P_0P}$ can be uniquely spanned by $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$. That is, there exists an unique $(\lambda_1, \lambda_2, ..., \lambda_d) \in k^d$ such that

$$\vec{P_0P} = \lambda_1\vec{P_0P_1} + ... + \lambda_d\vec{P_0P_d}.$$

This implies that

$$\begin{aligned} P = P_0 + \vec{P_0P} =& P_0 + \lambda_1\vec{P_0P_1} + ... + \lambda_d\vec{P_0P_d} \\ =&(1 - \sum_{i=1}^{d}\lambda_i)P_0 + \lambda_1(P_0 + \vec{P_0P_1}) + ... + \lambda_d(P_0 + \vec{P_0P_d}) \\ =&\lambda_0P_0 + \lambda_1P_1 + ... + \lambda_dP_d, \end{aligned}$$

where $\lambda_0 := (1 - \sum_{i=1}^{d}\lambda_i)$, and $(\lambda_0, \cdots, \lambda_d) \in k^{d+1}$ satisfies $\lambda_0 + ... + \lambda_d = 1$.

2. (a) Assume that $n < d = \dim_k V$, then $W = \langle\vec{P_0P_1}, \cdots, \vec{P_0P_n}\rangle$ is a proper subspace of V, then there exists such a vector $\vec{P_0P} \in V$, but $\vec{P_0P} \notin W$, i.e., there does not exist $(\lambda_1, ..., \lambda_n)$ such that $\vec{P_0P} = \sum_{i=1}^{n}\lambda_i\vec{P_0P_i}$. This would imply that for such a $P \in X$, there does not exist $(\lambda_0, ...\lambda_n) \in k^{n+1}$ with $\sum_{i=0}^{n}\lambda_i = 1$ such that $P = Bar(P_0, \cdots, P_d; \lambda_0, \cdots, \lambda_n)$. This contradicts with the assumption,

Since $\dim X = \dim V = d$, to simplify the proof, we can assume $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forms a basis of V.

(b) Assume that $d < n$, then it must be the case that $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ are linearly dependent. Since otherwise if they are not, then it implies that $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forms a basis of V and then $d = n$, contradicting our assumption $d < n$. Now since $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ are linearly dependent, there exists a unique tuple $(\lambda_1, \lambda_2, ..., \lambda_d)$, not all of them being zero, such that $\lambda_1\vec{P_0P_1} + ... + \lambda_d\vec{P_0P_d} = 0$. Assume $\lambda_i \neq 0$, then by dividing both sides of the equation by λ_i we have that $\vec{P_0P_i} = -\frac{\lambda_1}{\lambda_i}\vec{P_0P_1} - ... - \frac{\lambda_{i-1}}{\lambda_i}\vec{P_0P}_{i-1} - \frac{\lambda_{i+1}}{\lambda_i}\vec{P_0P}_{i+1} - \frac{\lambda_d}{\lambda_i}\vec{P_0P_d}$. This is equivalent to the statement that P_i can be uniquely represented as $P_i = (1+\sum_{j=1,j\neq j}^{d} \frac{\lambda_j}{\lambda_i})P_0 - \frac{\lambda_1}{\lambda_i}P_1 - ... - \frac{\lambda_{i-1}}{\lambda_i}P_{i-1} - \frac{\lambda_{i+1}}{\lambda_i}P_{i+1} - ... - \frac{\lambda_d}{\lambda_i}P_d$. But on the other hand we know that $P_i = P_i$, then, P_i has two different representations. This contradicts the hypothesis that any $P \in X$ can be represented uniquely as $Bar(P_0, \cdots, P_d; \lambda_0, \cdots, \lambda_n)$. Since both case (a) and case (b) can not happen, it must be the case that $n = d$. By our assumption $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forms a basis of V, then $P_0, \cdots, P_d$ are in general position.

3. If $(P_{\sigma(0)}, \cdots, P_{\sigma(d)})$ is in general position, then $(\vec{P_0P_1}, \cdots, \vec{P_0P_d})$ forms a basis of V. For any vector $\vec{P_0P} \in V$, there exists a unique tuple $(\lambda_1, ..., \lambda_d) \in k^d$ such that $\vec{P_0P} = \sum_{i=1}^{d} \lambda_i \vec{P_0P_i}$. This is equivalent to the statement that P can be written uniquely as $P = \sum_{i=0}^{d} \lambda_i P_i$ with $\sum_{i=0}^{d} \lambda_i = 1$. For any given permutation $\sigma \in S_{d+1}$, we have $P = \sum_{i=0}^{d} \lambda_i P_i = \sum_{i=0}^{d} \lambda_{\sigma(i)} P_{\sigma(i)}$, where $\sum_{i=0}^{d} \lambda_{\sigma(i)} = 1$. Then $P - P_{\sigma(0)} = \sum_{i=0}^{d} \lambda_{\sigma(i)} P_{\sigma(i)} - P_{\sigma(0)} = \sum_{i=0}^{d} \lambda_{\sigma(i)}(P_{\sigma(i)} - P_{\sigma(0)})$. This is $\vec{P_{\sigma(0)}P} = \sum_{i=0}^{d} \lambda_{\sigma(i)} \vec{P_{\sigma(0)}P_{\sigma(i)}}$, i.e., the vector $\vec{P_{\sigma(0)}P} \in V$ can be written uniquely as $\vec{P_{\sigma(0)}P} = \sum_{i=0}^{d} \lambda_{\sigma(i)} \vec{P_{\sigma(0)}P_{\sigma(i)}}$. Therefore $(P_{\sigma(0)}, \cdots, P_{\sigma(d)})$ is in general position.

Exercice 4. Soit X un espace affine de direction V. Soit $Y \subset X$ un sous-ensemble.

1. To show (a) implies (b), let $\{\vec{PP_1}, \cdots, \vec{PP_n}\}$ be a basis of W. Then for any $w \in W$, there exists $(\lambda_1, ..., \lambda_n) \in k^n$ such that $w = \sum_{i=1}^{n} \lambda_i \vec{PP_i}$. Then

$$\begin{aligned} Y =& \{P + \vec{w},\ \vec{w} \in W\} = \{P + \sum_{i=1}^{n} \lambda_i \vec{PP_i}, \lambda_i \in k\} \\ =& \{(1 - \sum_{i=i}^{n} \lambda_i)P + \sum_{i=1}^{n} \lambda_i P_i, \lambda_i \in k\} \\ =& \{Bar(P, \cdots, P_n; \lambda_0, \cdots, \lambda_n),\ \lambda_i \in k,\ \sum_{i=0}^{n} \lambda_i = 1\} \end{aligned}$$

Assume (b) is true, then we can write

$$\begin{aligned} Y =& \{\sum_{i=0}^{n} \lambda_i P_i,\ \lambda_i \in k,\ \sum_{i=0}^{n} \lambda_i = 1\} \\ =& \{P_0 + \sum_{i=0}^{n} \lambda_i (P_i - P_0),\ \lambda_i \in k\} \\ =& P_0 + \{\sum_{i=0}^{n} \lambda_i \vec{P_0P_i},\ \lambda_i \in k\} \\ =& P_0 + W, \end{aligned}$$

where $W = \langle \vec{P_0P_1}, \cdots, \vec{P_0P_n} \rangle$.

2. If $n > \dim_k W$, then the vectors $\{\vec{P_0P_1}, \cdots, \vec{P_0P_n}\}$ are linearly dependent over k, since otherwise they would form a basis of W and then we would have $n = \dim_k W$. Then there exists $(\lambda_1, ..., \lambda_n) \in k^n - \{(0, ..., 0)\}$ such that $\sum_{i=1}^n \lambda_i \vec{P_0P_i} = 0$. Assume without loss of generality that $\lambda_n \neq 0$, then $\vec{P_0P_n} = \sum_{i=1}^{n-1} \frac{-\lambda_i}{\lambda_n} \vec{P_0P_i}$. This implies that $W = \langle \vec{P_0P_1}, \cdots, \vec{P_0P_n} \rangle = \langle \vec{P_0P_1}, \cdots, \vec{P_0P_{n-1}} \rangle$. We can continue this process until we arrive at the case that $W = \langle \vec{P_0P_1}, \cdots, \vec{P_0P_d} \rangle$, where $d = \dim_k W$.

   If $\vec{P_0P_1}, \cdots, \vec{P_0P_n}$ forms a basis of W, then for any $P \in Y$, there exists an unique $(\lambda_1, ..., \lambda_n) \in k^n$ such that $\vec{P_0P} = \sum_{i=1}^n \lambda_i \vec{P_0P_i}$. This is equivalent to the uniquely representation of $P \in Y$ as the barycentre of $(P_0, \cdots, P_n) : P = (1 - \sum_{i=1}^n \lambda_i)P_0 + \sum_{i=1}^n \lambda_i P_i$. $(P_0, ..., P_n)$ forms an affine basis of Y.

3. This follows from the equivalent description in Part 1 and Part 2.