Solution suggestions série 7

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Exercice 4

Let X be an affine space with direction V. Let $AGL(X)$ be the group of affine automorphisms of X, $T(V)$ the subgroup of translations of $AGL(X)$ and

$$lin : AGL(X) \to GL(V)$$

be the group homomorphism 'linear part'. We know that $ker(lin) = T(V)$.(see Corollaire 3.2 in the lecture notes)

Qn. 2

Let $P \in X$ and

$$AGL(X)_P = \{\varphi \in AGL(X) |\, \varphi(P) = P\}$$

is the stabiliser of P. Define

$$lin_P : AGL(X)_P \to GL(V)$$

as the composition

$$AGL(X)_P \hookrightarrow AGL(X) \xrightarrow{lin} GL(V)$$

.

To show: lin_P is an isomorphism.
Proof
Injectivity: Left to the reader. See Lemme 3.1 in the lecture notes.

Surjectivity: Let $\varphi_o \in GL(V)$. Define $\varphi : X \to X$ by $\varphi(Q) = P + \varphi_0(v)$ $\forall Q \in X$. Since the map $V \to V$ given by $v \mapsto \varphi(P+v) - \varphi(P) = \varphi_0(v)$ is linear, φ is an affine transformation (by Théorème 3.1). Moreover $\varphi \in AGL(X)$ since it has the inverse $\psi : X \to X$ defined by $\psi(Q) = P + \varphi_0^{-1}(v)$ $\forall Q \in X$.
$AGL(X)_P$ is not normal:
Let $\varphi \in AGL(X)$. $\varphi AGL(X)_P \varphi^{-1} = AGL(X)_{\varphi(P)}$. So if $AGL(X)_P$ is normal, $AGL(X)_{\varphi(P)} \subseteq AGL(X)_P$ $\forall \varphi \in AGL(X)$. Since $T(V) \subseteq AGL(X)_P$, if $AGL(X)_P$ is normal, $AGL(X)_Q \subseteq AGL(X)_P$ $\forall Q \in X$. But if $dim(V) > 0$ for any two distinct points $P \neq Q$ (and indeed if $dim(V) > 0$ two distinct points can be found in X), we can find a $\varphi \in AGL(X)_Q$ s.t. $\varphi \notin AGL(X)_P$ which is a contradiction. (details left to the reader) $\square$

Qn 1

Let $P \in X$. A right inverse to lin is given by the following morphism

$$GL(V) \xrightarrow{lin_P^{-1}} AGL(X)_P \hookrightarrow AGL(X)$$

In words, if $\varphi_o \in GL(V)$, $lin_P^{-1}(\varphi_o) \in AGL(X)$ is a preimage of φ_o.

Qn 3

Let $X = V$, $GL(V) = AGL(X)_0$ is a subgroup of $AGL(X)$. (the equality follows from Corollaire 3.1 in the notes.)

Exercise 5

We wish to state and prove a general theorem from which we will deduce the statements in the exercise as corollaries. The slogan of the theorem is affine transformations are specified by specifying the image of any set of points in general position.

Theorem 1. *Let X and Y be affine spaces of directions V and W respectively. Let $P_0, ..., P_n$ be points in general position in X and $Q_0, ..., Q_n$ be arbitrary points in Y. There exists a unique affine transformation $\varphi : X \to Y$ s.t. $\varphi(P_i) = Q_i$ $\forall$ $0 \leq i \leq n$.*

Proof

Let P_i, Q_i be points in X and Y resp. satisfying the hypothesis of the theorem. Let $\varphi_0 : V \to W$ be the unique linear map s.t. $\varphi_0(P_i - P_0) = Q_i - Q_0$. Note that such a φ_0 exists and is unique since $\{P_i - P_0 | 1 \leq i \leq n\}$ form a basis of V. We define a map $\varphi : X \to Y$ by $\varphi(P_0 + v) := Q_0 + \varphi_0(v)$. Note that every point in X can be written as $P_0 + v$ for some unique v. Since $\varphi(P_0 + v) - \varphi(P_0) = \varphi_0(v)$ is linear, by Théorème 3.1, φ is an affine transformation. Moreover $\varphi(P_i) = \varphi(P_0 + (P_i - P_0)) = Q_0 + \varphi_0(P_i - P_0) = Q_0 + (Q_i - Q_0) = Q_i \ \forall\ 0 \leq i \leq n$.

Uniqueness

Since every point is a barycentre of the points P_i and affine transformations map barycentres to barycentres, it follows that there is a unique affine transformation satisfying the hypothesis of the theorem. □

Remark (**Hint for exercise 3**). In general, given points P_i and $Q_i \ \forall\ 0 \leq i \leq m$, the question of existence (resp. uniqueness) of affine transformation mapping P_i to $Q_i \ \forall\ 0 \leq i \leq m$ reduces to the question of existence (resp. uniqueness) of linear transformation from V to W mapping $P_i - P_0$ to $Q_i - Q_0$ $\forall\ 1 \leq i \leq m$.

Corollary. *Suppose an affine transformation $\varphi : X \to X$ satisfies $\varphi(P_i) = P_i$ $\forall\ 0 \leq i \leq n$ for points in general position. It follows that $\varphi = Id_X$.*

This corollary follows from the uniqueness part of the theorem.

Corollary. *Suppose in the theorem we also assume that $Q_0, ..., Q_n$ are in general position. It follows that φ is an isomorphism.*

Indeed, the inverse of φ is the unique affine transformation from Y to X that maps Q_i to $P_i \ \forall\ 0 \leq i \leq n$.

Now we proceed to deduce the solution to the questions from the theorem:

Qn 1

Let $\varphi \in AGL(X)$. Let $P_i \ \forall\ 0 \leq i \leq n$ be points in general position. We wish to show $\varphi(P_i) \ \forall\ 0 \leq i \leq n$ are in general position.

Let $Q \in X$ be a point. Since φ is surjective, $\exists\ P \in X$ s.t. $\varphi(P) = Q$. Let $P = Bar(P_0, ..., P_n, \lambda_0, ..., \lambda_n)$, it follows that $Q = Bar(\varphi(P_0), ..., \varphi(P_n), \lambda_0, ..., \lambda_n)$. Since $Q \in X$ was arbitrary, we conclude that $\varphi(P_i) \ \forall\ 0 \leq i \leq n$ forms a generating family. (see beginning of Série 7 for the definition.)

Further if $Bar(\varphi(P_0), ..., \varphi(P_n), \lambda_0, ..., \lambda_n) = Bar(\varphi(P_0), ..., \varphi(P_n), \mu_0, ..., \mu_n)$, $\varphi(Bar(P_0, ..., P_n, \lambda_0, ..., \lambda_n)) = \varphi(Bar(P_0, ..., P_n, \mu_0, ..., \mu_n))$. Since φ is injective we get $Bar(P_0, ..., P_n, \lambda_0, ..., \lambda_n) = Bar(P_0, ..., P_n, \mu_0, ..., \mu_n)$. We conclude that $\lambda_i = \mu_i \ \forall\ 0 \leq i \leq n$, using the fact that P_i are in general position. So we see that $\varphi(P_i) \ \forall\ 0 \leq i \leq n$ are free. (see beginning of Série 7 for the definition.)

We get by the above considerations an action of $AGL(X)$ on $A\mathcal{B}(X)$. $\square$

Qn 2

We wish to show that $A\mathcal{B}(X)$ is a principal homogeneous space of $AGL(X)$.

Transitivity: This follows from the second corollary of the theorem. **The action is free:** This follows from the first corollary of the theorem. $\square$