Solution suggestions série 7 continued

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Preliminary exercise

Let X and Y be affine spaces of directions V and W resp.

Question 1

Let $\{v_1, ..., v_n\}$ be a basis of V and $\{w_1, ..., w_n\}$ be arbitrary vectors in W.

T.S.: $\exists$ a unique $\varphi \in Hom_k(V, W)$ s.t. $\varphi_0(v_i) = w_i \ \forall \ 1 \leq i \leq n$

Let $v \in V$, we have $v = \sum c_i v_i$ where $c_i \in k \ \forall \ 1 \leq i \leq n$ are unique. Define $\varphi_0(v) = \sum c_i w_i$. We leave it to the reader to check φ_0 is linear. Uniqueness is clear from linearity.

Question 2

Let $P_0 \in X$, $Q_0 \in Y$ and $\varphi_0 \in Hom_k(V, W)$.

T.S. There exists a unique affine transformation $\varphi : X \to Y$ s.t. $\varphi(P_0) = Q_0$ and φ_0 is the linear part of φ.

Let $P \in X$, $P = P_0 + v$ for a unique $v \in V$. Define $\varphi(P) = Q_0 + \varphi_0(v)$. It follows from Théorème 3.1, that φ is an affine transformation. If $\psi : X \to Y$ is an affine transformation, we have $\psi(P) = \psi(P_0) + lin(\psi)(v)$. So an affine transformation is uniquely determined by the image of one point and the linear part.

Compare this question to the theorem in the solution of exercise 5.

Exercise 3

Let φ be the affine morphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ that maps

$$P_0 = (1,1,1),\ P_1 = (1,1,2),\ P_2 = (1,2,1),\ P_3 = (2,2,4)$$

to

$$Q_0 = (1,2,3),\ Q_1 = (1,3,3),\ Q_2 = (2,3,6),\ Q_3 = (1,2,4).$$

Question 1

We compute $v_i := P_i - P_0$

$$v_1 = (0,0,1),\ v_2 = (0,1,0),\ v_3 = (1,1,3)$$

These vectors form a basis of $\mathbb{R}^3$ i.e. the points P_i are in general position. It follows from the theorem in solution of exercise 5 that φ exists and is unique.

Question 2

We also compute $w_i := Q_i - Q_0$

$$w_1 = (0,1,0),\ w_2 = (1,1,3),\ w_3 = (0,0,1)$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map which satisfies $f(e_i) = v_i$ and let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map which satisfies $g(e_i) = w_i$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. It follows that $\varphi_0 = g \circ f^{-1}$. Note that the above statements use first question of the preliminary exercise. We made the above construction to calculate the matrix of φ_0 (w.r.t. the standard basis of $\mathbb{R}^3$). Here are the matrices of f, g, φ_0: (w.r.t the standard basis)

$$M_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}, M_f^{-1} = \begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, M_g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

and we get

$$M_{\varphi_0} = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 1 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$

The translation vector is given by $w = Q_0 - P_0 = (0, 1, 2)$. So $\varphi(P_0 + v) = P_0 + w + \varphi_0(v)$. The image of φ is $\mathbb{R}^3$ because φ_0 is surjective.

$\boldsymbol{\varphi^{-1}(3, 2, 1)}$:
$(3, 2, 1) - Q_0 = (2, 0, -2)$. Further $\varphi_0^{-1}(2, 0, -2) = (-8, -6, -26)$. We conclude that $\varphi^{-1}(3, 2, 1) = P_0 + (-8, -6, -26) = (-7, -5, -25)$.

Calculating in the same way as above $\varphi^{-1}(Q) = P_0 + \varphi_0^{-1}(Q - Q_0)$. Note that φ is invertible since it is surjective between affine space of same dimension or by observing that Q_i's are in general position and using the second corollary to our theorem. (solution of exercise 5) Also note:

$$M_{\varphi_0^{-1}} = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 0 & 1 \\ -10 & 1 & 3 \end{pmatrix}$$

Question 3

Now set

$$Q_0 = (1, 2, 3), Q_1 = (1, 3, 3),\ Q_2 = (1, 7/3, 10/3),\ Q_3 = (1, 2, 4).$$

There exists a unique φ mapping P_i to Q_i resp.(this only uses the fact that P_i are in general position). We list properties φ, calculations are left to the reader.

$$M_{\varphi_0} = \begin{pmatrix} 0 & 0 & 0 \\ -10/3 & 1/3 & 1 \\ 2/3 & 1/3 & 0 \end{pmatrix}, \text{Translation vector is } (0, 1, 2).$$

The image of φ is the 2 dimensional affine subspace given by $\{Q_0 + \lambda(0, 1, 0) + \mu(0, 0, 1) | \lambda, \mu \in \mathbb{R}\} = \{(1, 2 + \lambda, 3 + \mu) | \lambda, \mu \in \mathbb{R}\}$. In particular $\varphi^{-1}(3, 2, 1) = \phi$.

Question 4

Now consider

$$P_0 = (1, 1, 1),\ P_1 = (1, 1, 2),\ P_2 = (3/2, 3/2, 3),\ P_3 = (2, 2, 4)$$

to

$$Q_0 = (1,2,3),\ Q_1 = (1,3,3),\ Q_2 = (1,5/2,7/2),\ Q_3 = (1,2,4).$$

We want to understand if there exists an affine transformation that maps the P_i to Q_i and if it is unique. In the light of the preliminary exercise, this reduces to studying the existence and uniqueness of a linear map φ_0 which maps $v_i := P_i - P_0$ to $w_i := Q_i - Q_0$, $\forall\ 1 \leq i \leq 3$. We have,

$$v_1 = (0,0,1),\ v_2 = (1/2,1/2,2),\ v_3 = (1,1,3)$$

to

$$w_1 = (0,1,0),\ w_2 = (0,1/2,1/2),\ w_3 = (0,0,1).$$

Observe that $v_2 = \frac{1}{2}v_3 + \frac{1}{2}v_1$ and that v_1 and v_3 are linearly independent. (In other words v_1, v_2 and v_3 lie in a 2-dimensional subspace.) Moreover $w_2 = \frac{1}{2}w_3 + \frac{1}{2}w_1$, so any linear transformation that maps v_1 to w_1 and v_3 to w_3, automatically maps v_2 to w_2! Linear maps φ_0 with the above property therefore exist but are not unique. We proceed to describe the linear maps with this property, using first part of the preliminary exercise:

Let us consider as a basis for $\mathbb{R}^3$(Check!),

$$v_1' = (0,0,1),\ v_2' = (1,0,0),\ v_3' = (1,1,3)$$

we want to consider the linear transformation φ^w that maps the v_i's respectively to

$$w_1' = (0,1,0),\ w_2' = w := (c_1,c_2,c_3),\ w_3' = (0,0,1).$$

We can therefore parametrize all the linear maps which map v_1 to w_1 and v_3 to w_3 by φ^w for $w \in \mathbb{R}^3$. This is an affine subspace of $Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ (the vector space of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$). More explicitly, the linear transformations of interest are of the form $\varphi^w = \varphi^0 + c_1\psi^{e_1} + c_2\psi^{e_2} + c_3\psi^{e_3}$, $c_1, c_2, c_3 \in \mathbb{R}$ (i.e. the affine subspace $\varphi^0 + (\mathbb{R}\psi^{e_1} \oplus \mathbb{R}\psi^{e_2} \oplus \mathbb{R}\psi^{e_3})$) and ψ^w is the linear transformation that maps

$$v_1' = (0,0,1),\ v_2' = (1,0,0),\ v_3' = (1,1,3)$$

respectively to

$$w_1' = (0,0,0),\ w_2' = w,\ w_3' = (0,0,0).$$

and e_i is the ith standard basis vector

Any affine transformation that maps P_i to Q_i has linear part ϕ^w for some $w \in \mathbb{R}^3$ and maps P_0 to Q_0. □