EPFL Printemps 2019
Geometrie, MATH-125

Série 10: Correction

Exercise 1

1. This is similar to the proof of Proposition 3.15 in the course notes. We consider the characteristic polynomial of $M = M_\varphi$,

$$\begin{aligned} P_M(x) = \det(x \cdot Id_n - M) =& x^n - \operatorname{tr}(M)x^{n-1} + ... + (-1)^n \det(M) \\ =& x^n - \operatorname{tr}(M)x^{n-1} + ... + \det(M), \end{aligned}$$

where $(-1)^n = 1$ since n is even. Notice that $P_M(0) = \det(M) = -1 < 0$ by assumption, while $\lim_{x\to+\infty} P_M(x) = +\infty$ and $\lim_{x\to-\infty} P_M(x) = +\infty$. Then by the intermediate value theorem, $P_M(x)$ has a root $\lambda_1 \in (0,\infty)$ and a root $\lambda_2 \in (-\infty, 0)$. For $i = 1, 2$, since $\lambda_i Id_n - M$ are not invertible, there exist $\vec{v_i} \neq 0$ such that $\lambda_i \vec{v_i} - M \cdot \vec{v_i} = 0$. Then $\varphi(\vec{v_i}) = \lambda_i \vec{v_i}$. We have seen that (see Proposition 3.14) $\lambda_i = \pm 1$, in particular, $\lambda_i = 1$ and $\lambda_2 = -1$. Let $\mathbf{e}_1 = \vec{v_1}/\|\vec{v_1}\|$ and $\mathbf{e}_2 = \vec{v_2}/\|\vec{v_2}\|$, then we have $\varphi(\mathbf{e}_1) = \mathbf{e}_1$ and $\varphi(\mathbf{e}_2) = -\mathbf{e}_2$, as required. Moreover, $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular, since $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = \langle \varphi(\mathbf{e}_1), -\varphi(\mathbf{e}_2)\rangle = \langle \mathbf{e}_1, -\mathbf{e}_2\rangle$ which implies that $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = 0$.
2. We can choose a BON of the form $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, v_3, ..., v_n\}$, where $\{v_3, ..., v_n\}$ is a BON of $(\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_1)^\perp$. Then under $\mathcal{B}$, the matrix $M_{\varphi,\mathcal{B}}$ is of the form as required.

Exercise 2

2. Since $M_{B,\varphi} = (x_{ij})_{i,j}$ is an orthonormal matrix, each column vector $\vec{v_i}$ of $M_{B,\varphi}$ is of normal 1 : $\|\vec{v_i}\| = \sqrt{\sum_{j=1}^n x_{ij}^2} = 1$. In particular, each entry x_{ij} of the matrix $M_{B,\varphi}$ is of absolute value bounded by 1 : $|x_{ij}| \leqslant 1$. Hence $\operatorname{tr}(\varphi) = \sum_{i=1}^n x_{ii} \in [-n, n]$.

Exercise 3

1. Under the convenient orthonormal basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the matrix of φ is of the form
$$M_{\varphi,\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix},$$
where $c = \cos(\theta)$ and $s = \sin(\theta)$. From Exercise 2, we know that the trace and determinant of φ do not depend on the choices of the bases. Hence $\operatorname{tr}(\varphi) = \operatorname{tr}(M_{\varphi,\mathcal{B}}) = 1 + 2\cos(\theta) \in [-3, 3]$. If $\theta = \pi$, that is, if φ is an axial symmetry, then $\operatorname{tr}(\varphi) = 1 - 2 = -1$. If $\theta = 0$, i.e., $\varphi = id$, then $\operatorname{tr}(\varphi) = 3$.
2. Under the convenient orthonormal basis $\mathcal{B}$, the matrix of φ is of the form
$$M_{\varphi,\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, c = \cos(\theta), s = \sin(\theta).$$
Then $\operatorname{tr}(\varphi) = \operatorname{tr}(M_{\varphi,\mathcal{B}}) = -1 + 2\cos(\theta) \in [-3, 1]$. $\operatorname{tr}(\varphi) = -3$ if $\cos(\theta) = -1$, if $\theta = \pi$ and φ is a point symmetry (symetrie centrale). If $\operatorname{tr}(\varphi) = 1$, if $\cos(\theta) = 1$ and $\theta = 0$, in which case φ is an orthogonal symmetry with respect to the plane $\mathbb{R}\mathbf{e}_2 + \mathbb{R}\mathbf{e}_3$.
3. Under an appropriate orthonormal basis, the matrix of φ is of the form
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
which implies that $\operatorname{tr}(\varphi) = \operatorname{tr}(M) = 1$.
4. The matrix of a plane symmetry in the convenient orthonormal basis $\mathcal{B}$ is given by $M_{\varphi,\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let A be the base change matrix from $M_{\varphi,\mathcal{B}}$ to $M_{0,\varphi}$. Then $M_{0,\varphi} = A \cdot M_{\varphi,\mathcal{B}} \cdot A^{-1} = A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA$, where A is an orthogonal matrix. Now
$${}^tM_{0,\varphi} = {}^t(A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA) = {}^t({}^tA) \cdot {}^tM_{\varphi,\mathcal{B}} \cdot {}^tA = A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA = M_{0,\varphi},$$
since ${}^tM_{\varphi,\mathcal{B}} = M_{\varphi,\mathcal{B}}$.
5. (a) If $\varphi = id$ is the identity, then $\operatorname{tr}(\varphi) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$.

   (b) If φ is a point symmetry (symetrie centrale), then from Part 2, we know that $\operatorname{tr}(\varphi) = \operatorname{tr}(M_{\varphi,\mathcal{B}}) = -1 = 2\cos(\pi) = -3$ (with $\theta = \pi$ there).

(c) If φ is an axial symmetry, then from Part 1, we know that $\operatorname{tr}(\varphi) = \operatorname{tr}(M_{\varphi,\mathcal{B}}) = 1 + 2\cos(\pi) = -1$ (with $\theta = \pi$ there).

(d) If φ is a symmetry with respect to a plane, then from Part 3, we know that $\operatorname{tr}(\varphi) = \operatorname{tr}(M) = 1$.

6. If φ is a rotation, then from Part 1, we know that under the basis $\mathcal{B}$, $\operatorname{tr}(\varphi) = 1+2\cos(\theta)$ and $\det(\varphi) = 1$. Then $\frac{1}{2}(\operatorname{tr}(M_{\varphi,\mathcal{B}}) - \det(M_{\varphi,\mathcal{B}})) = \frac{1}{2}(1+2\cos(\theta)-1) = \cos(\theta)$.

   Similarly, if φ is an anti-rotation, then from Part 2, we know that $\operatorname{tr}(M_{\varphi,\mathcal{B}}) = -1 + 2\cos(\theta)$ and $\det(M_{\varphi,\mathcal{B}}) = -1$. Hence $\frac{1}{2}(\operatorname{tr}(M_{\varphi,\mathcal{B}}) - \det(M_{\varphi,\mathcal{B}})) = \frac{1}{2}(-1 + 2\cos(\theta) + 1) = \cos(\theta)$.

Exercise 6

1. In matrix notation we have

$$\varphi(x, y, z) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & -8 & 4 \\ 4 & 4 & 7 \\ -8 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

   The linear part is given by the matrix

$$\frac{1}{9}\begin{pmatrix} 1 & -8 & 4 \\ 4 & 4 & 7 \\ -8 & 1 & 4 \end{pmatrix}$$

   which is orthogonal of determinant 1 and trace 1, hence it defines a rotation of angle θ. The angle of rotation θ can be calculated as follows : by Exercise 2.6, $\cos(\theta) = \frac{1}{2}(\operatorname{tr}(M) - \det(M)) = \frac{1}{2}(1 - 1) = 0$, hence $\theta = \frac{\pi}{2}$.

   The axis of the rotation is given by $\ker(\varphi_0 - \mathrm{Id})$, the fixed point of φ_0. Since the vector $(-1, 2, 2) \in \ker(\varphi_0 - \mathrm{Id})$, we have a Vissage (see classiification 5.4.1 (3)) along the line $\mathbb{R}(-1, 2, 2)$.

2. Now we have

$$\psi(x, y, z) = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

   The linear part

$$\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

   is orthogonal and symmetric of determinant -1 and trace 1, and hence by Exercise 3.5, is a plane symmetry. The plane is defined to be $\ker(\psi_0 - \mathrm{Id})$, which is spanned by the two basis vectors $(1,1,0)$ and $(1,0,1)$. The translation vector $(1,-1,-1)$ is perpendicular to this plane, since it is perpendicular to both of these two vectors. Hence $(1,-1,-1) \in \ker^{\perp}(\psi_0 - \mathrm{Id}) = \mathrm{Im}(\psi_0 - \mathrm{Id})$. By the classification form 5.4.2 (2) ψ defines a symmetry with respect to a plane.

3. According to Exercise 5, the type of $\varphi \circ \psi \circ \varphi^{-1}$ is the same with the type of ψ, hence in our case , it is a symmetry with respect to a plane. In order to find this plane, note that the points that are fixed by $\varphi \circ \psi \circ \varphi^{-1}$ are exactly the points $\varphi(P)$, where P is a fixed point of ψ.

Exercise 7

1. In a positively oriented orthonormal basis $B = \{v_1, v_2, v_3\}$ where $\mathbb{R}v_1 = \mathbb{R}(1,1,1)$ is must be the rotation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/6) & -\sin(\pi/6) \\ 0 & \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

To find such a basis one normalizes v_1 to $v_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, then one finds an orthogonal unitary vector like $v_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and completes the basis with the cross product $v_3 = v_1 \times v_2 = (-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$. Another way is with the Gram–Schmidt process.

Finally, one computes the matrix in the canonical basis using the base change matrix $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. The matrix under the canonical base is therefore given by

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1}{3} & \frac{1+\sqrt{3}}{3} \end{pmatrix}.$$

2. The vector $(1,0,-1)$ is orthogonal to $(1,1,1) \in \ker(r - \mathrm{Id})$ and hence belongs to $\mathrm{Im}(r - \mathrm{Id})$. By the classification 5.4.1 (2) it is a rotation around the line of fixed points of angle $\pi/6$.

The axis of rotation of r' is equal to $D_{r'} = -z + D_0$ (see Proposition 3.16), where D_0 denotes the axis of rotation r, and z is defined to be a vector such that $(1,0,-1) = (r - \mathrm{Id})(z)$. For instance, one can take $z = (0, -1-\sqrt{3}, 1)$. Therefore the axis of rotation of r' is

$$\mathrm{Fix}(r') = -z + \mathbb{R}(1,1,1).$$

To calculate $(r')^{2018}$, since $2018 = 168 \times 12 + 2$, $(r')^{2018}$ gives 168 full rotations, followed by two rotations of $\pi/6$, which results in a rotation of $\pi/3$ around the axis described above.

3. This time $(2,2,2) \in \ker(r - \mathrm{Id})$. By the classification 5.4.1 (3), r'' is the composition of the affine rotation r around the axis $\mathbb{R}(1,1,1)$, followed by a translation with translation vector $(2,2,2)$, which is a vissage along $\mathbb{R}(1,1,1)$.
   Note that $t_{(2,2,2)} \circ r = r \circ t_{(2,2,2)}$ (see Theorem 3.11 (2)), then

$$(r'')^{2018} = (t_{(2,2,2)} \circ r)^{2018} = t_{(2,2,2)}^{2018} \circ r^{2018} = t_{(2,2,2)}^{2018} \circ r^{12 \cdot 168+2} = t_{(4036,4036,4036)} \circ r^2.$$