AN INTRODUCTION TO RIEMANNIAN GEOMETRY

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Contents

1. Introduction

Question 1.1. What kinds of quantities and operations appear in relation to analysis (or multivariable calculus) in a bounded open set $U \subset \mathbb{R}^n$?

Some possible answers:

- Functions: continuity, partial derivatives, integrals, L^p spaces, Taylor expansions, Fourier or related expansions
- Vector fields: gradient, curl, divergence
- Measures, distributions, flows
- Laplace operator, Laplace, heat and wave equations
- Integration by parts formulas (Gauss, divergence, Green)
- Tensor fields, differential forms
- Distance, distance-minimizing curves (line segments), area, volume, perimeter

Imagine similar concepts on a hypersurface (e.g. double torus in $\mathbb{R}^3$)

This course is an introduction to analysis on manifolds. The first part of the course title has the following Wikipedia description: "Mathematical Analysis is a branch of

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mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space)."

Following this description, our purpose will be to study in particular differentiation, integration, and differential equations on spaces that are more general than the standard Euclidean space $\mathbb{R}^n$. Different classes of spaces allow for different kinds of analysis:

- *Topological spaces* are a good setting for studying continuous functions and limits, but in general they do not have enough structure to allow studying derivatives
- The smaller class of *metric spaces* admits certain notions of differentiability, but in particular higher order derivatives are not always well defined
- *Differentiable manifolds* are modeled after pieces of Euclidean space and allow differentiation and integration, but they do not have a canonical Laplace operator and thus the theory of differential equations is limited

The class of spaces studied in this course will be that of Riemannian manifolds. These are differentiable manifolds with an extra bit of structure, a Riemannian metric, that allows to measure lengths and angles of tangent vectors. Adding this extra structure leads to a very rich theory where many different parts of mathematics come together. We mention a few related aspects, and some of these will be covered during this course (the more advanced topics that will be covered will be chosen according to the interests of the audience):

(1) *Calculus.* Riemannian manifolds are differentiable manifolds, hence the usual notions of multivariable calculus on differentiable manifolds apply (derivatives, vector and tensor fields, integration of differential forms)
(2) *Metric geometry.* Riemannian manifolds are metric spaces: there is a natural distance function on any Riemannian manifold such that the corresponding metric space topology coincides with the usual topology. Distances are realized by certain distinguished curves called geodesics, and these can be studied via a second order ODE (the geodesic equation).
(3) *Measure theory.* Any oriented Riemannian manifold has a canonical measure given by the volume form. The presence of this measure allows to integrate functions and to define L^p spaces on Riemannian manifolds.
(4) *Differential equations.* There is a canonical Laplace operator on any Riemannian manifold, and all the classical linear partial differential equations (Laplace, heat, wave) have natural counterparts
(5) *Dynamical systems.* The geodesic flow on a closed Riemannian manifold is a Hamiltonian flow on the cotangent bundle, and the geometry of the manifold is reflected in properties of the flow (such as complete integrability or ergodicity)

(6) *Conformal geometry.* The notions of conformal and quasiconformal mappings make sense on Riemannian manifolds, and there is enough underlying structure to provide many tools for studying them
(7) *Topology.* There are several ways of describing topological properties of the underlying manifold in terms of analysis. In particular, Hodge theory characterizes the cohomology of the space via the Laplace operator acting on differential forms, and Morse theory describes the topological type of the space via critical points of a smooth function on it
(8) *Curvature.* The notion of curvature is fundamental in mathematics, and Riemannian manifolds are perhaps the most natural setting for studying curvature. Related concepts include the Riemann tensor, the Ricci tensor, and scalar curvature. There has been recent interest in lower bounds for Ricci curvature and their applications
(9) *Inverse problems.* Many interesting inverse problems have natural formulations on Riemannian manifolds, such as integral geometry problems where one tries to determine a function from its integrals over geodesics, or spectral rigidity problems where one tries to determine properties of the underlying space from knowledge of eigenvalues of the Laplacian.
(10) *Geometric analysis.* There are many branches of mathematics that are called geometric analysis. One particular topic is that of geometric evolution equations, where geometric quantities evolve according to a certain PDE. One of the most famous such equations is Ricci flow, where a Riemannian metric is deformed via its Ricci tensor. This was recently used by Perelman to complete Hamilton's program for proving the Poincaré and geometrization conjectures.

2. Calculus in Euclidean spaces

Let U be any nonempty open subset of R^n (not necessarily bounded, and could be equal to R^n). We fix standard Cartesian coordinates $x = (x_1, \cdots, x_n)$ and will use these coordinates throughout this chapter. We may sometimes write x^j instead of x_j, and we will also denote by v_j or v^j the j-th coordinate of a vector $v \in \mathbb{R}^n$.

2.1. Functions and Taylor expansions.

2.1. Functions and Taylor expansions. Let $C(U)$ be the set of continuous functions on U. For partial derivatives, we will write

$$\partial_j f = \frac{\partial f}{\partial x_j} \quad \text{and} \quad \partial_{j_1 \cdots j_k} f = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}.$$

We denote by $C^k(U)$ the set of k times continuously differentiable real valued functions on U. Thus

$$C^k(U) = \Big\{ f \colon U \to \mathbb{R} : \partial_{j_1 \cdots j_l} f \in C(U) \text{ whenever } l \leq k \text{ and } j_1, \cdots, j_l \in \{1, \cdots, n\} \Big\}.$$

Recall also that if $f \in C^k(U)$, then $\partial_{j_1 \cdots j_k} f = \partial_{j_{\sigma(1)} \cdots j_{\sigma(k)}} f$ for any permutation σ of $\{1, \cdots, k\}$.

We also denote by $C^\infty(U)$ the infinitely differentiable functions on U, that is,

$$C^\infty(U) = \bigcap_{k\geq 0} C^k(U).$$

Theorem 2.1 (Taylor expansion)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{l=0}^{k} \frac{1}{l!}\Big[\sum_{j_1,\cdots,j_l} \partial_{j_1\cdots j_l} f(x_0)(x-x_0)_{j_1}\cdots(x-x_0)_{j_l}\Big] + R_k(x; x_0),$$

where $|R_k(x;x_0)| \leq \eta(|x-x_0|)|x-x_0|^k$ for some function η with $\eta(s) \to 0$ as $s \to 0$.

Remark 2.2. The Taylor expansion of order 2 is given by

$$f(x) = f(x_0) + \nabla f(x_0)\cdot(x-x_0) + \frac{1}{2}\nabla^2 f(x_0)(x-x_0)\cdots(x-x_0) + R_2(x;x_0),$$

where $\nabla f = (\partial_1 f, \cdots, \partial_n f)$ is the gradient of f and $\nabla^2 f(x) = \big(\partial_{jk} f(x)\big)_{j,k=1}^n$ is the Hessian matrix of f.

Proof. Considering $g(y) := f(x_0+y)$, we may assume that $x_0 = 0$. Assume that $B(0,r) \subset U$, fix $x \in B(0,r)$, and define

$$h\colon (-1-\varepsilon, 1+\varepsilon) \to \mathbb{R}, \quad h(t) := g(tx),$$

where $\varepsilon > 0$ satisfies $(1+\varepsilon)|x| < r$. Then h is a C^k function on $(-1-\varepsilon, 1+\varepsilon)$, and repeated use of the fundamental theorem of calculus gives

$$\begin{aligned}
h(t) &= h(t) - h(0) + h(0) = h(0) + \int_0^t h'(s)ds \\
&= h(0) + h'(0)t + \int_0^t (h'(s) - h'(0))ds = h(0) + h'(0)t + \int_0^t\int_0^s h''(u)duds \\
&= h(0) + h'(0)t + h''(0)\frac{t^2}{2} + \int_0^t\int_0^s (h''(u) - h''(0))duds \\
&= \cdots \\
&= \sum_{i=0}^k h^{(i)}(0)\frac{t^i}{i!} + \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \big(h^{(k)}(t_k) - h^{(k)}(0)\big)dt_k\cdots dt_1. \qquad (2.1)
\end{aligned}$$

Here we used that $\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} dt_k\cdots dt_1 = \frac{t^k}{k!}$ (exercise).

Now, computation shows

$$h'(t) = \partial_j f(tx)x_j, \quad h''(t) = \partial_{jl} f(tx)x_j x_l, \quad \cdots$$

and

$$h^{(k)}(t) = \partial_{j_1\cdots j_k} f(tx)x_{j_1}\cdots x_{j_k}.$$

Applying (2.1) with $t = 1$ gives the result in the theorem, where

$$R_k(x) = \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \Big[\partial_{j_1\cdots j_k} f(t_k x) - \partial_{j_1\cdots j_k} f(0)\Big]x_{j_1}\cdots x_{j_k}dt_k\cdots dt_1.$$

The bound for R_k follows since $\partial_{j_1\cdots j_k} f$ is uniformly continuous on compact sets. $\square$

At this point it may be good to mention another convenient form of the Taylor expansion, which we state but will not use. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers. Then $\mathbb{N}^n$ consists of all n-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$ where the α_j are nonnegative integers. Such an n-tuple is called a multi-index. We write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For partial derivatives, the notation

$$\partial^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$$

will be used. We also use the notation $\alpha! = \alpha_1! \cdots \alpha_n!$.

Theorem 2.3 (Taylor expansion, multi-index version)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_k(x; x_0),$$

where R_k satisfies similar bounds as before.

Proof. Exercise. $\square$

2.2. **Tensor fields.** If $f \in C^k(U)$, if $x \in U$ and if $v \in \mathbb{R}^n$ is such that $|v|$ is sufficiently small, we write the Taylor expansion given in Theorem 2.1 in the form

$$f(x+v) = \sum_{l=0}^{k} \frac{1}{l!}\Big[\sum_{j_1,\cdots,j_l=1}^{n} \partial_{j_1\cdots j_l} f(x) v_{j_1}\Big] + R_k(x+v; x).$$

The first few terms are

$$f(x+v) = f(x) + \partial_j f(x) v_j + \frac{1}{2}\partial_{jk} f(x) v_j v_k + \cdots$$

Looking at the terms of various degree motivates the following definition.

Definition 2.4 (Tensor fields)**.** An *m-tensor field* in U is a collection of functions $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$, where each $u_{j_1\cdots j_m}$ is in $C^\infty(U)$. The tensor field u is called *symmetric* if $u_{j_1\cdots j_m} = u_{j_{\sigma(1)}\cdots j_{\sigma(m)}}$ for any $j_1, \cdots, j_m$ and for any σ which is a permutation of $\{1, \cdots, m\}$.

Remark 2.5. This definition is specific to R^n, since we are deliberately not allowing any other coordinate systems than the Cartesian one. Later on we will consider tensor fields on manifolds, and their transformation rules under coordinate changes will be an important feature (these will decide whether the tensor field is covariant, contravariant or mixed). However, upon fixing a local coordinate system, all tensor fields will look essentially like the ones defined above.

Example 2.6. (1) The 0-tensor fields in U are just the scalar functions $u \in C^\infty(U)$

(2) The 1-tensor fields in U are of the form $u = (u_j)_{j=1}^n$, where $u_j \in C^\infty(U)$. Thus 1-tensor fields are exactly the vector fields in U; the tensor $(u_j)_{j=1}^n$ is identified with $(u_1, \cdots, u_n)$.

(3) The 2-tensor fields in U are of the form $u = \big(u_{j,k}\big)_{j,k=1}^n$, where $u_{jk} \in C^\infty(U)$. Thus 2-tensor fields can be identified with smooth matrix functions in U. The 2-tensor field is symmetric if the matrix is symmetric.

(4) If $f \in C^\infty(U)$, then we have for any $m \geq 0$ an m-tensor field $u = \big(\partial_{j_1\cdots j_m} f\big)_{j_1,\cdots,j_m=1}^n$ consisting of partial derivatives of f. This tensor field is symmetric since the mixed partial derivatives can be taken in any order.

Again by looking at the terms in the Taylor expansion, one can also think that an m-tensor $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$ acts on a vector $v \in \mathbb{R}^n$ by the formula

$$v \mapsto u_{j_1\cdots j_m}(x) v^{j_1} \cdots v^{j_m}.$$

The last expression can be interpreted as a multilinear map acting on the m-tuple of vectors $(v, \cdots, v)$.

Definition 2.7 (Multi-linear map)**.** If $m \geq 0$, an m-linear map is any map

$$L \colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$$

such that L is linear in each of its variables separately.

The following theorem is almost trivial, but for later purposes it will be good to know that a tensor field can be thought of in two ways: either as a collection of coordinate functions, or as a map on U that takes values in the set of multilinear maps.

Theorem 2.8 (Tensors as multilinear maps)**.** If $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$ is an m-tensor field on $U \subset \mathbb{R}^n$, then for any $x \in U$, there is an m-linear map $u(x)$ defined via

$$u(x)(v_1, \cdots, v_m) = u_{j_1\cdots j_m}(x) v_1^{j_1} \cdots v_m^{j_m}, \quad v_1, \cdots, v_m \in \mathbb{R}^n,$$

and it holds that $u_{j_1\cdots j_m}(x) = u(x)(e_{j_1}, \cdots, e_{j_m})$. Conversely, if T is a function that assigns to each $x \in U$ an m-linear map $T(x)$, and if the function $u_{j_1\cdots j_m} \colon x \mapsto T(x)(e_{j_1}, \cdots, e_{j_m})$ are in $C^\infty(U)$ for each $j_1, \cdots, j_m$, then $(u_{j_1\cdots j_m})$ is an m-tensor field in U.

Proof. Exercise. □

To get a picture of what Theorem 2.8 really says, we consider the case of 2-tensors. In this case, $u = (u_{jk})$ can be identified with matrix-valued functions $A = (a_{jk})_{j,k=1}^n$ with $a_{jk}(x) = u_{jk}(x)$. For each $x \in U$, we may regard $A(x)$ as a 2-linear map via

$$A(x)(v, w) = vA(x)w^T = \sum_{j,k=1}^n a_{jk}(x) v^j w^k.$$

It is clear that $A(x)$ is linear in both v and w, since $A(x)(av_1+v_2, w) = (av_1+v_2)A(x)w^T = aA(x)(v_1, w) + A(x)(v_2, w)$ and $A(x)(v, bw_1 + w_2) = vA(x)(bw_1 + w_2)^T = bA(x)(v, w_1) + A(x)(v, w_2)$ hold for each $a, b \in \mathbb{R}$.

2.3. **Vector fields and differential forms.** Let $U \subset \mathbb{R}^n$ be an open set. We wish to consider vector fields on U and certain operations related to vector fields.

Definition 2.9 (Vector fields)**.** A C^k *vector field* in U is a map $F = (F_1, \cdots, F_n): U \to \mathbb{R}^n$ such that all the component functions F_j are in $C^k(U)$. The set of vector fields on U is denoted by $C^k(U, \mathbb{R}^n)$.

Recall from Section 2.2 that vector fields are the same as 1-tensor fields. If $u \in C^\infty(U)$, the gradient of u gives rise to a vector field in U:

$$\mathrm{grad}\colon C^\infty(U) \to C^\infty(U, \mathbb{R}^n), \quad \mathrm{grad}(u) = (\partial_1 u, \cdots, \partial_n u).$$

If $F \in C^\infty(U, \mathbb{R}^n)$, the *divergence* of F gives rise to a function in U:

$$\mathrm{div}\colon C^\infty(U, \mathbb{R}^n) \to C^\infty(U), \quad \mathrm{div}(F) = \partial_1 F_1 + \cdots + \partial_n F_n.$$

The following basic identity suggests that in order to define the Laplace operator on a space, it may be enough to have a reasonable definition of divergence and gradient.

Lemma 2.10. $\mathrm{div} \circ \mathrm{grad} = \Delta$.

Proof. $\mathrm{div}\big(\mathrm{grad}(u)\big) = \partial_1(\partial_1 u) + \cdots + \partial_n(\partial_n u) = \Delta u.$ □

We will consider further operations on vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Curl in $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be open. If $F \in C^\infty(U, \mathbb{R}^2)$, the curl of F is the function

$$\mathrm{curl}(F) := \partial_1 F_2 - \partial_2 F_1.$$

Thus $\mathrm{curl}\colon C^\infty(U, \mathbb{R}^2) \to C^\infty(U)$.

Curl in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^3$ be open. If $F \in C^\infty(U, \mathbb{R}^3)$, the curl of F is the vector field

$$\mathrm{curl}(F) := \nabla \times F = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1).$$

Lemma 2.11. In two dimensions, one has

$$\mathrm{curl} \circ \mathrm{grad} = 0.$$

In three dimensions, one has

$$\mathrm{curl} \circ \mathrm{grad} = 0, \quad \mathrm{div} \circ \mathrm{curl} = 0.$$

Proof. If $U \subset \mathbb{R}^2$ and $u \in C^\infty(U)$, we have

$$\mathrm{curl}\big(\mathrm{grad}(u)\big) = \partial_1(\partial_2 u) - \partial_2\big(\partial_1 u\big) = 0.$$

If $U \subset \mathbb{R}^3$ and $u \in C^\infty(U)$, we have

$$\mathrm{curl}\big(\mathrm{grad}(u)\big) = \big(\partial_2\partial_3 u - \partial_3\partial_2 u, \partial_3\partial_1 u - \partial_1\partial_3 u, \partial_1\partial_2 u - \partial_2\partial_1 u\big) = 0.$$

Moreover, for $F \in C^\infty(U, \mathbb{R}^3)$ we have

$$\mathrm{div}\big(\mathrm{curl}(F)\big) = \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) = 0.$$

□

The previous lemma can be described in terms of two sequences: if $U \subset \mathbb{R}^2$ consider

$$C^\infty(U) \overset{\mathrm{grad}}{\to} C^\infty(U, \mathbb{R}^2) \overset{\mathrm{curl}}{\to} C^\infty(U) \tag{2.2}$$

and if $U \subset \mathbb{R}^3$ consider

$$C^\infty(U) \overset{\mathrm{grad}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\mathrm{curl}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\mathrm{div}}{\to} C^\infty(U). \tag{2.3}$$

In both sequences, the composition of any two subsequent operators is zero. This suggests that there may be further structure which underlies these situations and might extend to higher dimensions. This is indeed the case, and the calculus of differential forms (or exterior algebra) was developed to reveal this structure. We will next discuss this calculus in a simple case.

Differential forms. The purpose will be to rewrite for instance (2.3) as a sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U), \tag{2.4}$$

where $\Omega^k(U)$ will be the set of differential k-forms on $U \subset \mathbb{R}^3$, and d will be a universal operator that reduces to grad, curl, and div in the respective degrees.

Let $U \subset \mathbb{R}^n$ be open. Motivated by (2.2) and (2.3), we define

$$\Omega^0(U) := C^\infty(U)$$

and

$$\Omega^1(U) := C^\infty(U, \mathbb{R}^n).$$

Thus $\Omega^0(U)$ is the set of smooth functions in U, and any $\alpha \in \Omega^1(U)$ can be identified with a vector field $\alpha = (\alpha_j)_{j=1}^n$, where $\alpha_j \in C^\infty(U)$. We write formally

$$\alpha = (\alpha_j)_{j=1}^n = \alpha_j dx^j.$$

Remark 2.12. For the purposes of this section it is enough to think of dx^j as a formal object. However, the proper way to think of dx^j would be as a 1-form (the exterior derivative of the function $x^j \colon U \to \mathbb{R}$), i.e. as a map that assigns to each $x \in U$ the linear map $dx^j|_x \colon T_xU \to \mathbb{R}$ that satisfies $dx^j|_x(e_k) = \delta_k^j$, where $\{e_1, \cdots, e_n\}$ is the standard basis of $T_xU \approx \mathbb{R}^n$.

To define $\Omega^k(U)$ for $k \geq 2$, first define the set of ordered k-tuples

$$\mathcal{I}_k := \{(i_1, \cdots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

If $I \in \mathcal{I}_k$, we consider the formal object

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

Then $\Omega^k(U)$ will be thought of as the set

$$\Omega^k(U) = \{\alpha_I dx^I : \alpha_I \in C^\infty(U)\},$$

where the sum is over all $I \in \mathcal{I}_k$. The number of elements in $\mathcal{I}_k$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We can make the above formal definition rigorous.

Definition 2.13 (Differential form)**.** If $U \subset \mathbb{R}^n$, define for $0 \leq k \leq n$

$$\Omega^k(U) := C^\infty\left(U, \mathbb{R}^{\binom{n}{k}}\right).$$

The elements of $\Omega^k(U)$ are called *differential k-forms* on U, and any differential k-form $\alpha \in \Omega^k(U)$ can be written as

$$\alpha = (\alpha_I)_{I \in \mathcal{I}_k} = \alpha_I dx^I,$$

where $\alpha_I \in C^\infty(U)$ for each I.

Remark 2.14. Note that since $\binom{n}{k} = \binom{n}{n-k}$, the set $\Omega^{n-1}(U)$ can be identified with the set of vector fields on U, and $\Omega^n(U)$ with $C^\infty(U)$. In fact one has

$$\Omega^{n-1}(U) = \Big\{ \sum_{j=1}^{n} \alpha_j dx^1 \wedge \cdots \wedge \hat{dx^j} \wedge \cdots \wedge dx^n; \alpha_j \in C^\infty(U) \Big\}$$
$$\Omega^n(U) = \Big\{ f dx^1 \wedge \cdots \wedge dx^n; f \in C^\infty(U) \Big\},$$

where $\hat{dx^j}$ means that dx^j is omitted from the wedge product.

The above definition is correct, but to keep things simple we have avoided a detailed discussion of the *wedge product* $\wedge$. To define the d operator in (2.4) properly we need to say a little bit more. The wedge product is an associative product on elements of the form dx^I, satisfying

$$dx^j \wedge dx^k = -dx^k \wedge dx^j,$$

and more generally if $J = (j_1, \cdots, j_k)$ is a k-tuple, with $j_1, \cdots, j_k \in \{1, \cdots, n\}$ (not necessarily ordered), we should have

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k} = (-1)^{\mathrm{Sign}(\sigma)} dx^{j_{\sigma(1)}} \wedge \cdots \wedge dx^{j_{\sigma(k)}},$$

where σ is any permutation of $\{1, \cdots, k\}$. This implies two conditions:

- $dx^{j_1} \wedge \cdots \wedge dx^{j_k} = 0$ if $(j_1, \cdots, j_k)$ contains a repeated index
- $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ can be expressed as $\pm dx^I$ for a unique $I \in \mathcal{I}_k$ if $(j_1, \cdots, j_k)$ contains no repeated index.

With this understanding we make the following definition.

Definition 2.15 (Exterior derivative)**.** The *exterior derivative* is the map $d\colon \Omega^k(U) \to \Omega^{k+1}(U)$ defined by

$$d(\alpha_I dx^I) := \partial_j \alpha_I dx^j \wedge dx^I.$$

Example 2.16. (1) If $f \in \Omega^0(U)$ (so $f \in C^\infty(U)$), then df is the differential of f written as a 1-form:

$$df = \partial_j f dx^j.$$

(2) If $\alpha \in \Omega^1(U)$, say $\alpha = \alpha_k dx^k$ for some $\alpha_k \in C^\infty(U)$, then

$$d\alpha = \partial_j \alpha_k dx^j \wedge dx^k = \sum_{1 \le j < k \le n} (\partial_j \alpha_k - \partial_k \alpha_j) dx^j \wedge dx^k.$$

(3) Any $u \in \Omega^n(U)$ satisfies $du = 0$ since $dx^{j_1} \wedge \cdots \wedge dx^{j_{n+1}} = 0$ whenever $j_1, \cdots, j_{n+1} \in \{1, \cdots, n\}$ and there will be a repeated index.

The second example above gives an n-dimensional analogue of the curl operator, as also suggested by the following lemma:

Lemma 2.17 (The exterior derivatives in two and three dimensions)**.** (1) Let $U \subset \mathbb{R}^2$. If $f \in \Omega^0(U)$, then

$$df = (\mathrm{grad}(f))_j dx^j.$$

If $\alpha = F_1 dx^1 + F_2 dx^2 \in \Omega^1(U)$ and $F = (F_1, F_2)$, then

$$d\alpha = \big(\mathrm{curl}(F)\big) dx^1 \wedge dx^2.$$

(2) Let $U \subset \mathbb{R}^3$. If $f \in \Omega^0(U)$, then

$$df = \big(\mathrm{grad}(f)\big)_j dx^j.$$

If $\alpha = F_j dx^j \in \Omega^1(U)$ and $F = (F_1, F_2, F_3)$, then

$$d\alpha = \big(\mathrm{curl}(F)\big)_j dx^{\hat{j}},$$

where

$$dx^{\hat{1}} := dx^2 \wedge dx^3, dx^{\hat{2}} := dx^3 \wedge dx^1, \text{and } dx^{\hat{3}} := dx^1 \wedge dx^2.$$

Finally, if $u = F_j dx^{\hat{j}} \in \Omega^2(U)$ and $F = (F_1, F_2, F_3)$, then

$$du = \big((\mathrm{div}(F))\big) dx^1 \wedge dx^2 \wedge dx^3.$$

Proof. Exercise. □

Let us now verify that $d \circ d$ is always zero.

Lemma 2.18. $d \circ d = 0$ on $\Omega^k(U)$ for any k with $0 \le k \le n$.

Proof. If $\alpha = \alpha_I dx^I \in \Omega^k(U)$, then

$$d\alpha = \sum_{k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_k \alpha_I dx^k \wedge dx^I$$

and

$$d(d\alpha) = \sum_{j,k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_{jk} \alpha_I dx^j \wedge dx^k \wedge dx^I.$$

By the properties of the wedge product, we get

$$d(d\alpha) = \sum_{1 \le j < k \le n} \sum_{I \in \mathcal{I}_k} \big(\partial_{jk}\alpha_I - \partial_{kj}\alpha_I\big) dx^j \wedge dx^k \wedge dx^I,$$

which is zero since the mixed partial derivatives are equal. □

If $U \subset \mathbb{R}^n$ is open, we therefore have a sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \tag{2.5}$$

and the composition of any two subsequent operators is zero. This gives the desired generalization of (2.2) and (2.3) to any dimension. In fact we have obtained much more: as we will see during this course, differential forms turn out to be an object of central importance in many kinds of of analysis on manifolds.

Differential forms as tensors. It will be useful to intepret differential forms as tensor fields satisfying an extra condition.

Definition 2.19 (Alternating tensor field)**.** An m-tensor field $\big(u_{j_1 \cdots j_m}\big)_{j_1, \cdots, j_m = 1}^{n}$ in $U \subset \mathbb{R}^n$ is called *alternating* if $u_{j_{\sigma(1)} \cdots j_{\sigma(m)}} = (-1)^{\mathrm{Sign}(\sigma)} u_{j_1 \cdots j_m}$ for any $j_1, \cdots, j_m$ and for any σ which is a permutation of $\{1, \cdots, m\}$.

We understand that 0-tensor fields and 1-tensor fields are always alternating. A 2-tensor field $u = (u_{jk})_{j,k=1}^n$ is alternating if and only if $u_{kj} = -u_{jk}$ for any j, k, i.e. the matrix (u_{jk}) is skew-symmetric at each point. An m-tensor field $u = (u_{j_1\cdots j_m})$ is alternating if and only if $u_{j_1\cdots j_m}$ changes sign when any two indices are interchanged (since any permutation can be expressed as the product of transpositions). Note that for an alternating tensor, $u_{j_1\cdots j_m} = 0$ whenever $(j_1, \cdots, j_m)$ contains a repeated index.

Theorem 2.20. If $U \subset \mathbb{R}^n$ is open and $0 \le k \le n$, the set $\Omega^k(U)$ can be identified with the set of alternating k-tensor fields on U.

Proof. Consider the map

$$T\colon \Omega^k(U) \to \{\text{alternating } k\text{-tensors}\}, \quad \alpha dx^I \mapsto \left(\tilde{\alpha}_{j_1\cdots j_k}\right),$$

where

$$\tilde{\alpha}_{j_1\cdots j_k} := \begin{cases} 0, \ (j_1, \cdots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}}(-1)^{\mathrm{Sign}(\sigma)}\alpha_I, \ (j_1, \cdots, j_k) \text{ contains no repeated index.} \end{cases}$$

Here, σ is the permutation of $\{1, \cdots, k\}$ such that $I = (j_{\sigma(1)}, \cdots, j_{\sigma(k)})$ is the unique element of $\mathcal{I}_k$ containing the same entries as $(j_1, \cdots, j_k)$. The constant $\frac{1}{\sqrt{k!}}$ is a harmless normalizing factor which will be useful later. Then $\tilde{\alpha}_{j_1\cdots j_k}$ is alternating by construction. It is clear that T is injective, and surjectivity follows since any alternating tensor is uniquely determined by the elements $\tilde{\alpha}_I$ where $I \in \mathcal{I}_k$. $\square$

Cohomology. By Lemma (2.18), we observe that

$$u = d\alpha \text{ for some } \alpha \in \Omega^{k-1}(U) \Rightarrow du = 0.$$

This may be rephrased as follows:

$$\mathrm{Im}\left(d|_{\Omega^{k-1}(U)}\right) \text{ is a linear subspace of } \ker\left(d|_{\Omega^k(U)}\right).$$

We express this in one more way: if $u \in \Omega^k(\Omega)$, we say that u is closed if $du = 0$ and that u is exact if $u = d\alpha$ for some $\alpha \in \Omega^{k-1}(U)$. Thus, any exact differential form is closed. The question of whether any closed form is exact depends on the topological properties of U. To study this property we make the following definition.

Definition 2.21 (de Rham cohomology)**.** The de Rham cohomology groups of U are defined by

$$H^k_{\mathrm{dR}}(U) = \ker\left(d|_{\Omega^k(U)}\right) / \,\mathrm{Im}\left(d|_{\Omega^{k-1}(U)}\right), \quad 0 \le k \le n.$$

By this definition each $H^k_{\mathrm{dR}}(U)$ is in fact a (quotient) vector space, not just a group. Recall that

Any closed k-form is exact if and only if $H^k_{\mathrm{dR}}(U) = \{0\}$. This happens for all $k \ge 1$ at least when U has very simple topology.

Lemma 2.22 (Poincaré lemma)**.** If $U \subset \mathbb{R}^n$ is open and star-shaped with respect to some $x_0 \in U$ (meaning that for any $x \in U$ the line segment between x_0 and x lies in U), then

$$H^k_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R}, \ k = 0, \\ \{0\}, \ 1 \le k \le n. \end{cases}$$

Proof. For simplicity we only do the proof for $n = 2$, see [8] for the general case (which is somewhat more involved). Assume that U is star-shaped with respect to 0. We have

$$H^0_{\mathrm{dR}}(U) = \ker \big(d|_{\Omega^0(U)}\big) = \big\{f \in C^\infty(U), \mathrm{grad}(f) = 0\big\}.$$

Since U is connected and star-shaped with respect to 0, $\nabla f = 0$ on U implies that $f \equiv f(0)$ is constant. Thus $H^0_{\mathrm{dR}}(U)$ is one-dimensional and isomorphic to $\mathbb{R}$.

We next show that $H^1_{\mathrm{dR}}(U) = \{0\}$, that is, for any $F \in C^\infty(U, \mathbb{R}^2)$, we have

$$\mathrm{curl}(F) = 0 \Rightarrow F = \mathrm{grad}(f) \text{ for some } f \in C^\infty(U).$$

Let $F = (F_1, F_2)$ satisfy $\partial_1 F_2 - \partial_2 F_1 = 0$. Then f should be some kind of integral of F, in fact we may just take

$$f(x) := \int_0^1 F_j(tx)x^j dt, \quad x \in U.$$

Since $\partial_1 F_2 = \partial_2 F_1$, we have

$$\begin{aligned} \partial_1 f(x) &= \int_0^1 \big[\partial_1 F_j(tx)tx^j + F_1(tx)\big]dt \\ &= \int_0^1 \big[\partial_1 F_1(tx)tx^1 + \partial_2 F_1(tx)tx^2 + F_1(tx)\big]dt \\ &= \int_0^1 \frac{d}{dt}\big[tF_1(tx)\big]dt = F_1(x). \end{aligned}$$

Similarly, $\partial_2 f(x) = F_2(x)$, showing that $F = \mathrm{grad}(f)$.

Finally, we show that $H^2_{\mathrm{dR}}(U) = \{0\}$, which means that

$$f \in C^\infty(U) \Rightarrow f = \mathrm{curl}(F) \quad \text{for some } F \in C^\infty(U, \mathbb{R}^2).$$

As in the previous case, F_j should be integrals of f. We may define

$$F_1(x) := -\int_0^1 f(tx)tx_2 dt \quad \text{and} \quad F_2(x) := \int_0^1 f(tx)tx_1 dx.$$

Then

$$\begin{aligned} \partial_1 F_2 - \partial_2 F_1 &= \int_0^1 \big[\partial_1 f(tx)t^2 x_1 + \partial_2 f(tx)t^2 x_2 + 2tf(tx)\big]dt \\ &= \int_0^1 \frac{d}{dt}\big[t^2 f(tx)\big]dt = f(x). \end{aligned}$$

□

We conclude by mentioning some facts about the de Rham cohomology groups (for more details see [8]):

- The de Rham cohomology groups are *topological invariants*: if U and V are homeomorphic open sets in Euclidean space, then $H^k_{\mathrm{dR}}(U)$ and $H^k_{\mathrm{dR}}(V)$ are isomorphic as vector spaces for each k. This gives a potential way of showing that two sets U and V are not homeomorphic; it would be enough to check that some cohomology groups are not isomorphic
- Note however that it is possible for non-homeomorphic spaces to have the same cohomology groups
- In many cases (e.g. if $U \subset \mathbb{R}^n$ is a bounded open set with nice boundary), the vector spaces $H^k_{\mathrm{dR}}(U)$ are finite dimensional. The dimension of $H^k_{\mathrm{dR}}(U)$ is a known topological invariant, namely the *k-th Betti number* of U.
- Very loosely speaking, the cohomology groups may give some information about "holes" in a set. For instance, if $K_1, \cdots, K_N$ are disjoint closed balls in $\mathbb{R}^n$, then

$$H^k_{\mathrm{dR}}\big(\mathbb{R}^n \setminus \cup_j^N K_j\big) = \begin{cases} \mathbb{R}, \text{if } k = 0, \\ \mathbb{R}^N, \text{if } k = n-1, \\ \{0\}, \text{otherwise} \end{cases}$$

Later in this course we will discuss Hodge theory, which studies the cohomology groups $H^k_{\mathrm{dR}}(M)$ where M is a compact manifold via the Laplace operator acting on differential forms on M.

2.4. **Riemannian metrics.** An open set $U \subset \mathbb{R}^n$ is often thought to be "homogeneous" (the set looks the same near every point) and "flat" (if U is considered as a subset of $\mathbb{R}^{n+1}$ lying in the hyperplane $\{x_{n+1} = 0\}$, then U has the geometry induced by the flat hypersurface $\{x_{n+1} = 0\}$. In this section, we will introduce extra structure on U which makes it "inhomogeneous" (the properties of the set vary from point to point) and "curved" (U has some geometry that is different from the geometry induced by a flat hypersurface $\{x_{n+1} = 0\}$).

Motivation. An intuitive way of introducing this extra structure is to think of U as a medium where sound waves propagate. The properties of the medium are described by a function $c\colon U \to \mathbb{R}_+$, which is thought of as the sound speed of the medium. If U is homogeneous, the sound speed is constant ($c(x) = 1$ for each $x \in U$), but if U is inhomogeneous, then the sound speed varies from point to point.

Consider now a C^1 curve $\gamma\colon [0,1] \to U$. The tangent vector $\dot\gamma(t)$ of this curve is thought to be a vector at the point $\gamma(t)$. If the sound speed is constant ($c \equiv 1$), the length of the tangent vector is just the Euclidean length:

$$\Big|\dot\gamma(t)\Big|_e := \Big[\sum_{j=1}^n \dot\gamma^j(t)^2\Big]^{\frac{1}{2}}.$$

In case of a general sound speed $c\colon U \to \mathbb{R}_+$, one can think that at points where c is large the curve moves very quickly and consequently has short length. Thus we may define the length of $\dot{\gamma}(t)$ with respect to the sound speed c by

$$\Big|\dot{\gamma}(t)\Big|_c := \frac{1}{c(\gamma(t))}\Big[\sum_{j=1}^{n}\dot{\gamma}^j(t)^2\Big]^{\frac{1}{2}}.$$

It is useful to generalize the above setup in two directions. First, in addition to measuring lengths of tangent vectors we would also like to measure angles between tangent vectors (in particular we want to know when two tangent vectors are orthogonal). Second, if the sound speed is a scalar function on U, then the length of a tangent vector is independent of its direction (the medium is *isotropic*). We wish to allow the medium to be *anisotropic*, which will mean that the sound speed may depend on direction and should be a matrix valued function.

In order to measures lengths and angles of tangent vectors, it is enough to introduce an inner product on the space of tangent vectors at each point. The tangent space is defined as follows:

Definition 2.23 (Tangent space)**.** If $U \subset \mathbb{R}^n$ is open and $x \in U$, the *tangent space* at x is defined as

$$T_xU := \{x\} \times \mathbb{R}^n.$$

The *tangent bundle* of U is the set

$$TU := \bigcup_{x\in U} T_xU.$$

Of course, each T_xU can be identified with $\mathbb{R}^n$ (and we will often do so), and a vector $v \in T_xU$ is written in terms of its coordinates as $v = (v^1, \cdots, v^n)$. Now if $\langle\cdot,\cdot\rangle$ is any inner product on $\mathbb{R}^n$, there is some positive definite symmetric matrix $A = (a_{jk})_{j,k=1}^n$ such that

$$\langle v, w\rangle = Av \cdot w, \quad v, w \in \mathbb{R}^n.$$

(The proof is left as an exercise, hint: take $a_{jk} = \langle e_j, e_k\rangle$) The next definition introduces an inner product on the space of tangent vectors at each point:

Definition 2.24 (Riemannian metric)**.** A *Riemannian metric* on U is a matrix-valued function $g = (g_{jk})_{j,k=1}^n$ such that each g_{jk} is in $C^\infty(U)$, and $(g_{jk}(x))$ is a positive definite symmetric matrix for each $x \in U$. The corresponding *inner product* on T_xU is defined by

$$\langle v, w\rangle_g := g_{jk}(x)v^jw^k, \quad v, w \in T_xU.$$

The *length* of a tangent vector is

$$|v|_g := \langle v, v\rangle_g^{1/2} = (g_{jk}(x)v^jv^k)^{1/2}, \quad v \in T_xU.$$

The *angle* between two tangent vectors $v, w \in T_xU$ is the number $\theta_g(v, w) \in [0, \pi]$ defined by

$$\cos\theta_g(v, w) = \frac{\langle v, w\rangle_g}{|v|_g|w|_g}.$$

We will often drop the subscript and write $\langle\cdot,\cdot\rangle$ or $|\cdot|$ if the metric g is fixed. To connect the above definition to the discussion about sound speeds, a scalar sound speed

$c(x)$ corresponds to the Riemannian metric

$$g_{jk}(x) = \frac{1}{c(x)^2}\delta_{jk}.$$

Finally, we introduce some notation that will be very useful.

Notation. If $g = (g_{jk})$ is a Riemannian metric on U, we write

$$(g^{jk})_{j,k=1}^n = g^{-1}$$

for the inverse matrix of $(g_{jk})_{j,k=1}^n$, and

$$|g| = \det(g)$$

for the determinant of the matrix $(g_{jk})_{j,k=1}^n$. In particular, we note that $g_{jk}g^{kl} = \delta_j^l$ for any $j, l = 1, \cdots, n$.

2.5. **Geodesics. Lengths of curves**. Consider an open set U that is equipped with a Riemannian metric g. As we saw above, one can measure lengths of tangent vectors with respect to g, and this makes it possible to measure lengths of curves as well.

Definition 2.25 (Regular curve and its length)**.** A smooth map $\gamma\colon [a,b] \to U$ whose tangent vector $\dot\gamma(t)$ is always nonzero is called a *regular curve.* The *length* of γ is defined by

$$L_g(\gamma) := \int_a^b |\dot\gamma(t)|_g dt.$$

The length of a piecewise regular curve is defined as the sum of lengths of the regular parts.

The *Riemannian distance* between two points $p, q \in U$ is defined by

$$d_g(p,q) := \inf\Big\{L_g(\gamma); \gamma\colon [a,b] \to U \text{ is piecewise regular with } \gamma(a) = p \text{ and } \gamma(b) = q\Big\}.$$

Since we only use the given Riemannian metric g on U, we will often omit the sub/superscript g in the corresponding quantity.

Fact. $L(\gamma)$ is independent of the way the curve γ is parametrized, and that we may always parametrize γ by *arc-length* so that $|\dot\gamma(t)| = 1$ for all t. (Proof is left as an exercise)

The previous exercise shows that we can always reparametrize a piecewise regular curve γ by arc length, so that one will have $|\dot\gamma(t)| = 1$ for all t. A curve satisfying $|\dot\gamma(t)| \equiv 1$ is called a *unit speed curve* (similarly a curve satisfying $|\dot\gamma(t)| \equiv$constant is called a *constant speed curve*).

Geodesic equation. We now wish to show that any length minimizing curve satisfies a certain ordinary differential equation.

Theorem 2.26 (Length minimizing curves are geodesics)**.** Suppose $U \subset \mathbb{R}^n$ is open, let g be a Riemannian metric on U, and let $\gamma\colon [a,b] \to U$ be a piecewise regular unit speed curve. Assume that γ minimizes the distance between its endpoints, in the sense that

$$L(\gamma) \le L(\eta)$$

for any piecewise regular curve η from $\gamma(a)$ to $\gamma(b)$. Then γ is a regular curve, and it satisfies the geodesic equation

$$\ddot{\gamma}^l(t) + \Gamma^l_{jk}\big(\gamma(t)\big)\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0, \quad 1 \leq l \leq n, \tag{2.6}$$

where Γ^l_{jk} are the Christoffel symbols of the metric g:

$$\Gamma^l_{jk} := \frac{1}{2} g^{lm}\big(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}\big), \quad 1 \leq j,k,l \leq n.$$

Example 2.27. If g is the Euclidean metric on U, so that $g_{jk}(x) = \delta_{jk}$, then all the Christoffel symbols Γ^l_{jk} are zero. The geodesic equation becomes just

$$\ddot{\gamma}^l(t) = 0, \quad 1 \leq l \leq n.$$

Solving this equation shows that

$$\gamma(t) = tv + w$$

for some vectors $v, w \in \mathbb{R}^n$. Thus Theorem 2.26 recovers the classical fact that any length minimizing curve in Euclidean space is a line segment.

Any smooth curve that satisfies the geodesic equation (2.6) is called a *geodesic*, and the conclusion of Theorem 2.26 can be rephrased so that any length minimizing curve is a geodesic. The fact that length minimizing curves satisfy the geodesic equation gives powerful tools for studying these curves. For instance, one can show that

- any geodesic has constant speed and is therefore regular
- given any $x \in U$ and $v \in T_x U$, there is a unique geodesic starting at point x in direction v
- any geodesic minimizes length at least locally (but not always globally)
- a set U with Riemannian metric g is *geodesically complete*, meaning that every geodesic is defined for all $t \in \mathbb{R}$, if and only if the metric space (U, d_g) is complete (this is the *Hopf-Rinow theorem*).

Variations of curves. Let $\gamma\colon [a,b] \to U$ be a piecewise regular length minimizing curve. We will prove Theorem 2.26 by considering families of curves (γ_s) where $s \in (-\varepsilon, \varepsilon)$ and $\gamma_0 = \gamma$, and all curves γ_s start at $\gamma(a)$ and end at $\gamma(b)$. Such a family is called a *variation* (or a *fixed-endpoint variation*) of γ. By the length minimizing property,

$$L(\gamma_0) \leq L(\gamma_s) \quad \text{ for } s \in (-\varepsilon, \varepsilon),$$

so if the dependence on s is at least C^1 we obtain that $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$. This fact, applied to many different families γ_s, will imply that γ is smooth and solves the geodesic equation.

If (γ_s) is a family of curves with $\gamma_0 = \gamma$, we think of $V(t) := \frac{\partial}{\partial s}\gamma_s(t)|_{s=0}$ as the "infinitesimal variation" of the curve γ that leads to the family (γ_s). The vector $V(t)$ should be thought of as an element of $T_{\gamma(t)}U$. The next result shows that one can reverse this process, and obtain a variation of γ from any given infinitesimal variation V.

In this result and below, we assume that the piecewise regular curve γ is fixed and that there is a subdivision of $[a,b]$,

$$a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b,$$

such that the curves $\gamma|_{(t_j,t_{j+1})}$ is regular for each j with $0 \le j \le N$.

Lemma 2.28 (Variations of curves)**.** If $V\colon [a,b] \to \mathbb{R}^n$ is a continuous map such that $V|_{(t_j,t_{j+1})}$ is C^∞ for each j and $V(a) = V(b) = 0$, then there exists $\varepsilon > 0$ and a continuous map

$$\Gamma\colon (-\varepsilon,\varepsilon) \times [a,b] \to U$$

such that the curves $\gamma_s\colon [a,b] \to U$, $\gamma_s(t) := \Gamma(s,t)$ satisfying the following

- each γ_s is a piecewise regular curve with endpoints $\gamma(a)$ and $\gamma(b)$, and $\gamma_s|_{(t_j,t_{j+1})}$ is regular for each j,
- $\gamma_0 = \gamma$,
- $s \mapsto \gamma_s(t)$ is C^∞ and $\frac{d}{ds}\gamma_s(t)|_{s=0} = V(t)$ for each $t \in [a,b]$.

Proof. Define

$$\Gamma\colon (-\varepsilon,\varepsilon) \times [a,b] \to U, \quad \Gamma(s,t) := \gamma(t) + sV(t),$$

where ε is so small that Γ takes values in U. The properties follow immediately from the definition. $\square$

We can now compute the derivative $\frac{d}{ds}L(\gamma_s)|_{s=0}$ that was mentioned above. In classical terminology, this is called the *first variation* of the length functional.

Lemma 2.29 (First variation formula)**.** Let γ be a piecewise regular unit speed curve, and let (γ_s) be a variation of γ associated with V as in Lemma 2.28. Then

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = -\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle D_t\dot\gamma(t), V(t)\rangle dt - \sum_{j=1}^{N}\langle \Delta\dot\gamma(t_j), V(t_j)\rangle,$$

where $D_t\dot\gamma(t)$ is the element of $T_{\gamma(t)}U$ defined by

$$\big(D_t\dot\gamma(t)\big)^l := \ddot\gamma^l(t) + \Gamma^l_{jk}\big(\gamma(t)\big)\dot\gamma^j(t)\dot\gamma^k(t), \quad 1 \le l \le n,$$

and $\Delta\dot\gamma(t_j) := \dot\gamma(t_j+) - \dot\gamma(t_j-)$ is the jump of $\dot\gamma(t)$ at t_j.

Remark 2.30. We will later give an invariant meaning to $D_t\dot\gamma(t)$ and interpret it as the covariant derivative of $\dot\gamma(t)$ along the curve γ. However, at this point it is enough to think of $D_t\dot\gamma(t)$ just as some expression that comes out when we compute the derivative $\frac{d}{ds}L(\gamma_s)|_{s=0}$.

Proof. Define

$$I(s) := L(\gamma_s) = \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \Big[g_{pq}\big(\gamma_s(t)\big)\dot\gamma_s^p(t)\dot\gamma_s^q(t)\Big]^{\frac{1}{2}} dt.$$

To prepare for computing the derivative $I'(0)$, define two vector fields

$$T(t) := \partial_t\gamma_s(t)|_{s=0} = \dot\gamma(t), \; V(t) := \partial_s\gamma_s(t)|_{s=0}.$$

Since $|\dot\gamma_0(t)| = |T(t)| \equiv 1$ and (g_{jk}) is symmetric, we have

$$I'(0) = \frac{1}{2}\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \Big(\partial_r g_{pq}(\gamma(t))V^r(t)T^p(t)T^q(t) + 2g_{pq}(\gamma(t))\dot V^p(t)T^q(t)\Big)dt.$$

Recall the following integration by parts formula:

$$\int_a^b f(t)h'(t)e(t)dt = \big(f(t)h(t)e(t)\big)\Big|_{t=a}^{t=b} - \int_a^b \big(f'(t)h(t)e(t) + f(t)h(t)e'(t)\big)dt.$$

Applying it to the last term above with $f(t) = g_{pq} \circ \gamma(t), h(t) = V^p(t)$ and $e(t) = T^q(t)$, we obtain that

$$\begin{aligned} I'(0) = \sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} &\Big[\frac{1}{2}\partial_r g_{pq}(\gamma)T^pT^q - \partial_m g_{rq}(\gamma)T^mT^q - g_{rq}(\gamma)\dot{T}^q\Big]V^r dt \\ &+ \sum_{j=0}^{N}\Big[\langle V(t_{j+1}), T(t_{j+1}+)\rangle - \langle V(t_j), T(t_j-)\rangle\Big]. \end{aligned}$$

Using that $V(t_0) = V(t_{N+1}) = 0$ and that V is continuous, the boundary term becomes $-\sum_{j=1}^N \langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle$ as required. For the integrals, we use that

$$\partial_m g_{rq}(\gamma)T^mT^q = \frac{1}{2}\Big(\partial_m g_{rq}(\gamma) + \partial_q g_{rm}(\gamma)\Big)T^mT^q,$$

which gives

$$\begin{aligned} -\langle D_t\dot{\gamma}(t), V(t)\rangle &= -g_{rq}(\gamma)\big(\dot{T}^q + \Gamma^q_{jk}T^jT^k\big)V^r \\ &= \Big(-g_{rq}(\gamma)\dot{T}^q + \frac{1}{2}\big[\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}\big]T^jT^k\Big)V^r \\ &= \Big(-g_{rq}(\gamma)\dot{T}^q + \frac{1}{2}\big[\big(\partial_m g_{rq}(\gamma) + \partial_q g_{rm}(\gamma)\big)T^mT^q - \partial_r g_{pq}T^pT^q\big]\Big)V^r \\ &= \Big(-g_{rq}(\gamma)\dot{T}^q - \frac{1}{2}\partial_r g_{pq}T^pT^q + \partial_m g_{rq}T^mT^q\Big)V^r. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.26. Let $\gamma\colon [a,b] \to U$ be a piecewise regular unit speed curve that minimizes the length between its endpoints. If V is any vector field as in Lemma 2.28 and (γ_s) is the corresponding variation of γ, we must have

$$L(\gamma_0) \le L(\gamma_s)$$

for $s \in (-\varepsilon, \varepsilon)$. Therefore, $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$. The first variation formula, Lemma 2.29, then shows that

$$\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle D_t\dot{\gamma}(t), V(t)\rangle dt + \sum_{j=1}^{N}\langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle = 0 \tag{2.7}$$

for any such V.

We first show that γ solves the geodesic equation on each interval (t_j, t_{j+1}). Fix $j \in \{0, \cdots, N\}$ and choose V such that

$$V(t) := \varphi(t)D_t\dot{\gamma}(t),$$

where φ is any function in $C_0^\infty\big((t_j,t_{j+1})\big)$. This V is an an admissible choice in Lemma 2.29 and (2.7) implies that

$$\int_{t_j}^{t_{j+1}} \varphi(t)|D_t\dot\gamma(t)|^2 dt = 0$$

for any $\varphi \in C_0^\infty\big((t_j,t_{j+1})\big)$. Thus we must have $D_t\dot\gamma(t)|_{(t_j,t_{j+1})} = 0$ for each j.

We next show that γ has no corners and is a C^1 curve in $[a,b]$. Going back to (2.7), we have

$$\sum_{j=1}^{N} \langle \Delta\dot\gamma(t_j), V(t_j)\rangle = 0$$

for any V with $V(a) = V(b) = 0$. Now if $\Delta\dot\gamma(t_j) \neq 0$ for some j, then we can choose V with $V(t_j) = \Delta\dot\gamma(t_j)$ and $V(t_k) = 0$ for $k \neq j$. This implies that

$$|\Delta\dot\gamma(t_j)|^2 = 0,$$

which contradicts the assumption $\Delta\dot\gamma(t_j) \neq 0$. This shows that we must have $\Delta\dot\gamma(t_j) = 0$ for each j, and it follows that $\gamma \in C^1\big([a,b]\big)$.

Finally, since $\gamma|_{(t_j,t_{j+1})}$ solves the geodesic equation for each j and since γ is C^1 near each t_j, the existence and uniqueness of ODE implies that $\gamma|_{(t_j,t_{j+1})}$ is the unique smooth continuation of the solution $\gamma|_{(t_{j-1},t_j)}$. Thus in fact γ solves the geodesic equation and is smooth near each t_j, and γ is a regular curve solving the geodesic equation on $[a,b]$. □

The previous proof shows actually more than stated in the theorem. We say that a piecewise regular curve γ is a *critical point* of the length functional L if $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$ for any fixed-endpoint variation of γ as in Lemma 2.28.

Theorem 2.31. The critical points of L are exactly the geodesic curves.

Proof. The proof of Theorem 2.26 shows that any critical point of L is a geodesic curve. To see the converse, let γ be a geodesic curve so that γ is C^∞ and $D_t\dot\gamma(t) = 0$ in $[a,b]$. By the first variation formula, Lemma 2.29, any such curve satisfies $\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = 0$, so any geodesic must be a critical point of L. □

Remark 2.32. Let us give a more geometric interpretation of the proof of Theorem 2.26. Suppose that γ is a piecewise regular curve which is smooth in (t_j,t_{j+1}) for $0 \leq j \leq N$. The preceding proof shows that

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = -\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle D_t\dot\gamma(t), V(t)\rangle dt - \sum_{j=1}^{N}\langle \Delta\dot\gamma(t_j), V(t_j)\rangle,$$

where (γ_s) is a variation of γ related to V as in Lemma 2.29. Choosing

$$V(t) := \varphi(t) D_t\dot\gamma(t),$$

where φ is a nonnegative function supported in (t_j,t_{j+1}) shows that

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = -\int_{t_j}^{t_{j+1}} \varphi(t)|D_t\dot\gamma(t)|^2 dt \leq 0.$$

Thus if $D_t\dot\gamma(t)\neq 0$ somewhere in (t_j,t_{j+1}), the derivative can be made strictly negative. This means we can always make the curve γ shorter by deforming it in the direction of $D_t\dot\gamma(t)$.

Assume now that γ solves the geodesic equation (2.6) in each segment (t_j,t_{j+1}) where it is smooth. If one has $\Delta\dot\gamma(t_j)\neq 0$ and if we choose V so that $V(t_j)=\Delta\dot\gamma(t_j)$ and $V(t_k=0)$ for $k\neq j$, then

$$\frac{d}{ds}L(\gamma_s)|_{s=0}=-|\Delta\dot\gamma(t)|<0.$$

This shows that a "broken geodesic" with corner at t_j can always be made shorter by deforming it in the direction of $\Delta\dot\gamma(t_j)$. This argument of "rounding the corner" was the key point in showing that length minimizing curves are C^∞.

2.6. **Integration and inner products.** This section will largely consist of definitions. We explain a natural way of integrating functions with respect to a Riemannian metric g, given by the volume form dV_g. This leads to an L^2 inner product first for scalar functions and then for vector fields and tensor fields. Finally we discuss the codifferential operator δ, which is the adjoint of the exterior derivative of d with respect to the L^2 inner product on differential forms. On 1-forms δ can be interpreted as a Riemannian divergence operator. The operator δ will be used in the next section to define the Laplace operator.

Integration. Let U be an open set, and let g be a Riemannian metric on U. If f is a function in (say) $C_c(U)$, we wish to consider the integral of f over U with respect to the metric g. The idea is that the metric g gives a way of measuring infinitesimal volumes, in the same way that it allows to measure lengths and angles of tangent vectors.

Motivation. Since in this chapter we are restricting ourselves to using Cartesian coordinates, the integral of f over U should be approximately given by

$$\int_U f(x)d\,\mathrm{Vol}_g\approx\sum_{j=1}^N f(x_j)\,\mathrm{Vol}_g(Q_j), \tag{2.8}$$

where $\{Q_1,\cdots,Q_N\}$ are very small congruent cubes whose sides are parallel to the Cartesian coordinate axes such that the cubes approximately tile U, and x_j is the center of Q_j. Now if Q_j has sidelength h, one should have

$$\mathrm{Vol}_g(Q_j)=\mathrm{Vol}_g\left(he_1|_{x_j},\cdots,he_n|_{x_j}\right),$$

where $\mathrm{Vol}_g(v_1,\cdots,v_n)$ is the Riemannian volume of the parallelepiped generated by the v_j (this is the set $\{\sum_{j=1}^n t_jv_j : t_j\in[0,1]\}$).

The volume should have the following properties if the v_j have very small (infinitesimal) length:

(a): If $v_1,\cdots,v_n$ are orthogonal with respect to g, one should have

$$\mathrm{Vol}_g(v_1,\cdots,v_n)\approx|v_1|_g\cdots|v_n|_g$$

(b): If A is a matrix with $Av_j=\lambda_jv_j$, $j=1,\cdots,n$, one should have

$$\mathrm{Vol}_g(Av_1,\cdots,Av_n)\approx\lambda_1\cdots\lambda_n\,\mathrm{Vol}_g(v_1,\cdots,v_n)$$

(c): More generally if A is any $n \times n$ matrix, then one should have

$$\mathrm{Vol}_g(Av_1, \cdots, Av_n) \approx \det(A)\,\mathrm{Vol}_g(v_1, \cdots, v_n)$$

Fix now a point $x \in U$, write $G = (g_{jk}(x))_{j,k=1}^n$, and note that the set $\{G^{-1/2}e_1, \cdots, G^{-1/2}e_n\}$ is an g-orthonormal basis of T_xU:

$$\begin{aligned}\langle G^{-1/2}e_j, G^{-1/2}e_k\rangle_g &= g_{pq}(x)\big(G^{-1/2}e_j\big)^p\big(G^{-1/2}e_k\big)^q = G(G^{-1/2}e_j)\cdot(G^{-1/2}e_k)\\ &= G^{-1/2}GG^{-1/2}e_j\cdot e_k = e_j\cdot e_k = \delta_{jk}.\end{aligned}$$

Thus the volume of an infinitesimal parallelepiped should be

$$\begin{aligned}\mathrm{Vol}_g\left(he_1|_x, \cdots, he_n|_x\right) &\approx h^n\,\mathrm{Vol}_g\left(G^{1/2}(G^{-1/2}e_1)|_x, \cdots, G^{1/2}(G^{-1/2}e_n)|_x\right)\\ &\approx h^n|g(x)|^{1/2},\end{aligned}$$

where $|g(x)| = \det(g_{jk}(x))$. Going back to (2.8), this would give

$$\int_U f(x)d\,\mathrm{Vol}_g(x) \approx \sum_{j=1}^N f(x_j)|g(x_j)|^{1/2}h^n \overset{h\to 0}{\to} \int_U f(x)|g(x)|^{1/2}dx.$$

The above discussion motivates the following definitions:

Definition 2.33 (Riemannian volume and integration)**.** Let $U \subset \mathbb{R}^n$ be open, and let g be a Riemannian metric on U. If $f \in C_c(U)$, we define the integral of f on U by

$$\int_U f(x)d\,\mathrm{Vol}_g(x) := \int_U f(x)|g(x)|^{1/2}dx.$$

The *Riemannian volume* of a measurable set $E \subset U$ is

$$\mathrm{Vol}_g(E) := \int_E |g(x)|^{1/2}dx.$$

If $1 \le p < \infty$, the L^p norm of f is

$$\|f\|_{L^p(U,dV_g)} := \Big(\int_U |f|^p dV_g\Big)^{1/p},$$

where for notational simplicity, we write V_g for Vol_g. The space $L^p(U, dV_g)$ is the completion of $C_c(U)$ in the L^p norm. It is easy to show that $L^p(U, dV_g)$ is a Banach space whenever $1 \le p < \infty$.

Remark 2.34. The quantity dV_g is usually called the *volume form* of the Riemannian manifold (U, g). To justify this terminology, one should interpret dV_g as the differential n-form (element of $\Omega^n(U)$) given by

$$dV_g = |g|^{1/2}dx^1 \wedge \cdots \wedge dx^n.$$

One can equivalently think of dV_g as a measure, i.e. (using the Riesz representation theorem for measures) as a linear operator acting on functions in $C_c(U)$ by

$$f \mapsto \int_U f dV_g.$$

In the present setting where $U \subset \mathbb{R}^n$, this measure is absolutely continuous with respect to Lebesgue measure ($dV_g(x) = |g(x)|^{1/2}dx$).

Inner products on L^2. The most important case of L^p spaces during this course is $p = 2$. In fact, $L^2(U, dV_g)$ is a Hilbert space with the following inner product.

Definition 2.35. If $u, v \in L^2(U, dV_g)$, we define

$$(u, v)_{L^2} := \int_U uv dV_g.$$

We now wish to define an L^2 inner product for vector fields and tensor fields on U as well. The case of vector fields comes naturally: if $F, G \in C_c(U, \mathbb{R}^n)$ are two vector fields, so that $F(x), G(x) \in T_xU$ for each $x \in U$, the g-inner product of $F(x)$ and $G(x)$ is

$$\langle F(x), G(x)\rangle_g = g_{jk}(x)F^j(x)G^k(x). \tag{2.9}$$

The L^2-inner product of F and G is then defined by

$$\begin{aligned}(F, G)_{L^2} &:= \int_U \langle F(x), G(x)\rangle_g dV_g(x)\\ &= \int_U g_{jk}(x)F^j(x)G^k(x)|g(x)|^{1/2}dx.\end{aligned}$$

Next consider the case of 1-forms. Let α and β be two 1-forms in U whose coordinate functions are in $C_c(U)$, meaning that $\alpha = \alpha_j dx^j$ and $\beta = \beta_k dx^k$, where $\alpha_k, \beta_k \in C_c(U)$. If $\alpha(x)$ denotes the expression $\alpha_j(x)dx^j$, in analogy with (2.9) it seems natural to define the g-inner product

$$\begin{aligned}(\alpha, \beta)_{L^2} &:= \int_U \langle \alpha(x), \beta(x)\rangle_g dV_g(x)\\ &= \int_U g^{jk}(x)\alpha_j(x)\beta_k(x)|g(x)|^{1/2}dx. \end{aligned}\tag{2.10}$$

Motivated by (2.10), one can define the L^2 inner product of two tensor fields with components in $C_c(U)$. In particular, this gives an L^2 inner product on differential forms since k-forms can be identified with certain (alternating) k-tensor fields by Theorem 2.20.

Definition 2.36. Let $u = \left(u_{j_1\cdots j_m}\right)_{j_1,\cdots,j_m=1}^n$ and $v = \left(v_{k_1\cdots k_m}\right)_{k_1,\cdots,k_m=1}^n$ be two tensor fields such that each $u_{j_1\cdots j_m}$ and $v_{k_1\cdots k_m}$ is in $C_c(U)$. The L^2 inner product of u and v is

$$(u, v)_{L^2} := \int_U g^{j_1k_1}(x)\cdots g^{j_mk_m}(x)u_{j_1\cdots j_m}v_{k_1\cdots k_m}|g(x)|^{1/2}dx.$$

If α and β are differential k-forms whose component functions are in $C_c(U)$, we denote by

$$(\alpha, \beta)_{L^2} := (\tilde{\alpha}, \tilde{\beta})_{L^2}$$

the inner product of the corresponding tensor fields as in Theorem 2.20.

Recall that if $\alpha = \alpha_I dx^I$ is a k-form, Theorem 2.20 identifies α with the k-tensor $\tilde{\alpha}$ defined by

$$\tilde{\alpha}_{j_1\cdots j_k} := \begin{cases} 0, (j_1, \cdots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}}\varepsilon_{j_1\cdots j_k}\alpha_{R(j_1,\cdots,j_k)}, (j_1, \cdots, j_k) \text{ contains no repeated index,} \end{cases}$$

where $R(j_1, \cdots, j_k) = (j_{\sigma(1)}, \cdots, j_{\sigma(k)})$ and σ is the unique permutation of $\{1, \cdots, k\}$ such that $j_1 < j_2 < \cdots < j_k$ (thus R puts the indices in inreasing order) and $\varepsilon_{j_1,\cdots,j_k} = (-1)^{\mathrm{Sign}(\sigma)}$.

Notice that if α and β are 1-forms, this inner product is equal to (2.10).

Example 2.37. Let $U \subset \mathbb{R}^n$ be open and let g be the Euclidean metric, so $g_{jk} = \delta_{jk}$. Then $|g(x)| = 1$ and $g^{jk} = \delta^{jk}$. If $\alpha = \alpha_j dx^j$ and $\beta = \beta_k dx^k$ are two 1-forms with $\alpha_j, \beta_k \in C_c(U)$, and if $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$ and $\vec{\beta} = (\beta_1, \cdots, \beta_n)$ are the corresponding vector fields, then

$$(\alpha, \beta)_{L^2} = \int_U \sum_{j=1}^n \alpha_j \beta_j dx = \int_U \vec{\alpha} \cdot \vec{\beta} dx.$$

Moreover, if $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$ and $v = \big(v_{k_1\cdots k_m}\big)_{k_1,\cdots,k_m=1}^n$ are two vector fields with components in $C_c(U)$, then

$$(u, v)_{L^2} = \int_U \sum_{j_1,\cdots,j_m=1}^n u_{j_1\cdots j_m} v_{j_1\cdots j_m} dx.$$

Codifferential. Our next purpose is to consider the exterior derivative $d\colon \Omega^k(U) \to \Omega^{k+1}(U)$ and to compute its formal adjoint operator in the L^2 inner product on forms. Below, we write $\Omega_c^k(U)$ for the set of compactly supported k-forms in U (thus $\alpha = \alpha_I dx^I$ is in $\Omega_c^k(U)$ if $\alpha_I \in C_c^\infty(U)$ for each I).

Theorem 2.38 (Codifferential)**.** Let $U \subset \mathbb{R}^n$ be open and let g be a Riemannian metric on U. For each k with $0 \le k \le n$, there is a unique linear operator

$$\delta\colon \Omega^k(U) \to \Omega^{k-1}(U)$$

having the property

$$(d\alpha, \beta)_{L^2} = (\alpha, \delta\beta)_{L^2}, \quad \alpha \in \Omega_c^{k-1}(U), \beta \in \Omega^k(U). \tag{2.11}$$

The operator δ satisfies $\delta \circ \delta = 0$ and $\delta|_{\Omega^0(U)} = 0$. It is a linear first order differential operator acting on component functions, and on 1-forms it is given by

$$\delta\beta := -|g|^{-1/2}\partial_j\big(|g|^{1/2} g^{jk}\beta_k\big), \beta = \beta_k dx^k \in \Omega^1(U). \tag{2.12}$$

The proof is based on the integration by parts formula

$$\int_U u(\partial_j v) dx = -\int_U (\partial_j u) v dx, \quad u \in C^1(U),\ v \in C_c^1(U). \tag{2.13}$$

Proof. We begin with the case $k=1$. Let $\beta=\beta dx\in\Omega^k(U)$. To compute $\delta\beta$ satisfying (2.11), we take $\alpha\in\Omega^0_c(U)=C^\infty_c(U)$ and compute

$$\begin{aligned}\langle d\alpha,\beta\rangle&=\int_U\langle d\alpha,\beta\rangle dV_g=\int_U g^{jk}\partial_j\alpha\beta_k|g|^{1/2}dx\\&=-\int_U\alpha|g|^{-1/2}\partial_j\big(|g|^{1/2}g^{jk}\beta_k\big)dV_g.\end{aligned}$$

Thus (2.11) will be satisfied for $k=1$ if we define $\delta\colon\Omega^1(U)\to\Omega^0(U)$ by (2.12).

Let us now show that for any k, there is an operator $\delta\colon\Omega^k(U)\to\Omega^{k-1}(U)$ such that (2.11) holds. Let $\alpha\in\Omega^{k-1}_c(U)$ and $\beta\in\Omega^k(U)$. Using the definitions and integration by parts, we obtain

$$\begin{aligned}\langle d\alpha,\beta\rangle&=\int_U\langle\partial_i\alpha_I dx^i\wedge dx^I,\beta_J dx^J\rangle_g dV_g\\&=\int_U(\partial_i\alpha_I)\beta_J\langle dx^i\wedge dx^I,dx^J\rangle_g|g|^{1/2}dx\\&=-\int_U\alpha_I|g|^{-1/2}\partial_i\Big[|g|^{1/2}\langle dx^i\wedge dx^I,dx^J\rangle_g\beta_J\Big]dV_g.\end{aligned}$$

Write $\gamma^I:=-|g|^{-\frac12}\partial_i\Big[|g|^{1/2}\langle dx^i\wedge dx^I,dx^J\rangle_g\beta_J\Big]$. It follows that

$$\langle d\alpha,\beta\rangle_{L^2}=\int_U\alpha_I\gamma^I dV_g.$$

We wish to find $\gamma=\gamma_L dx^L\in\Omega^{k-1}(U)$ such that $\alpha_I\gamma^I=\langle\alpha,\gamma\rangle_g$. This can be done by *lowering indices*. First let $\tilde\alpha=(\tilde\alpha_{i_1\cdots i_{k-1}})$ and $\tilde\gamma=(\tilde\gamma^{i_1\cdots i_{k-1}})$ be the alternating tensor fields corresponding to α_I and γ^I, so for instance $\tilde\gamma^{i_1\cdots i_{k-1}}:=\frac{1}{\sqrt{(k-1)!}}\varepsilon^{i_1\cdots i_{k-1}}\gamma^{R(i_1,\cdots,i_{k-1})}$. Let

$$\tilde\gamma_{l_1\cdots l_{k-1}}:=g_{l_1i_1}\cdots g_{l_{k-1}i_{k-1}}\tilde\gamma^{i_1\cdots i_{k-1}}$$

and let $\gamma=\gamma_L dx^L$ be the $(k-1)$-form corresponding to $\tilde\gamma$. Then

$$\begin{aligned}\langle\alpha,\gamma\rangle_g=\langle\tilde\alpha,\tilde\gamma\rangle_g&=g^{i_1l_1}\cdots g^{i_{k-1}l_{k-1}}\tilde\alpha_{i_1\cdots i_{k-1}}\big[g_{l_1p_1}\cdots g_{l_{k-1}p_{k-1}}\tilde\gamma^{p_1\cdots p_{k-1}}\big]\\&=\tilde\alpha_{i_1\cdots i_{k-1}}\tilde\gamma^{i_1\cdots i_{k-1}}=\frac{1}{(k-1)!}\alpha_{R(i_1\cdots i_{k-1})}\gamma^{R(i_1\cdots i_{k-1})}=\alpha_I\gamma^I.\end{aligned}$$

Combining the above arguments, we have proved that

$$(d\alpha,\beta)_{L^2}=(\alpha,\gamma)_{L^2}$$

for all $\alpha\in\Omega^{k-1}_c(U)$. Here $\gamma\in\Omega^{k-1}(U)$ is determined uniquely by this identity, thus setting $\delta\beta:=\gamma$ satisfies (2.12). Insepcting the above argument shows that $\delta\beta=\gamma_L dx^L$, where for $L=(l_1,\cdots,l_{k-1})$,

$$\gamma_L=-g_{l_1i_1}\cdots g_{l_{k-1}i_{k-1}}|g|^{-\frac12}\partial_i\big[|g|^{1/2}\langle dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}},dx^J\rangle_g\beta_J\big].$$

Thus δ is a first order operator acting on the component functions β_J.

It is clear that $\delta|_{\Omega^0(U)}=0$, and the condition $\delta\circ\delta=0$ follows from (2.11) and the fact that $d\circ d=0$. $\square$

If $U \subset \mathbb{R}^n$ is an open set, in Section 2.3, we studied the sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U), \tag{2.14}$$

where $d \circ d = 0$. This sequence does not depend on any Riemannian metric on U. However, if we introduce a Riemannian metric g on U, then Theorem 2.38 shows that there is another sequence

$$\Omega^0(U) \xleftarrow{\delta} \Omega^1(U) \xleftarrow{\delta} \cdots \xleftarrow{\delta} \Omega^{n-1}(U) \xleftarrow{\delta} \Omega^n(U), \tag{2.15}$$

where $\delta \circ \delta = 0$. As we will explain later, the sequences (2.14) and (2.15) and the corresponding cohomology groups turn out to be dual to each other: this is related to *Poincaré duality.*

2.7. **Laplace-Beltrami operator.** In this section we will see that on any open set equipped with a Riemannian metric, there is a canonical second order elliptic operator, called the *Laplace-Beltrami operator*, which is an analogue of the usual Laplacian in $\mathbb{R}^n$.

Motivation. Let first U be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and consider the Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{2.16}$$

Solutions of the equation $\Delta u = 0$ are called *harmonic functions*, and by standard results for elliptic PDE, for any $f \in H^1(U)$, there is a unique solution $u \in H^1(U)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ u = f & \text{on } \partial U. \end{cases} \tag{2.17}$$

The last line means that $u - f \in H_0^1(U)$.

One way to produce the solution of (2.17) is based on variational methods and Dirichlet's principle (see [5]). We define the Dirichlet energy

$$E(v) := \frac{1}{2} \int_U |\nabla v|^2 dx, \quad v \in H^1(U).$$

If we define the admissible class

$$\mathcal{A}_f := \{v \in H^1(U) : v = f \text{ on } \partial U\},$$

then the solution of (2.17) is the unique function $u \in \mathcal{A}_f$ which minimizes the Dirichlet energy:

$$E(u) \le E(v) \quad \text{for all } v \in \mathcal{A}_f.$$

The heuristic idea is that the solution of (2.17) represents a physical system in equilibrium, and therefore should minimize a suitable energy functional. The point is that one can start from the energy functional $E(\cdot)$ and conclude that any minimizer u must satisfy $\Delta u = 0$, which gives another way to define the Laplace operator.

From this point on, let $U \subset \mathbb{R}^n$ be open and let g be a Riemannian metric on U. Although there is no immediately obvious analogue of (2.16) that would take into account the metric g, there is a natural analogue of the Dirichlet energy. It is given by

$$E(v) := \frac{1}{2}\int_U |dv|^2 dV, \quad v \in H^1(U).$$

Here $|dv|$ is the Riemannian length of the 1-form dv, and dV is the volume form.

We wish to find a differential equation which is satisfied by minimizers of $E(\cdot)$. Suppose $u \in H^1(U)$ is a minimizer which satisfies $E(u) \le E(u+t\varphi)$ for all $t \in \mathbb{R}$ and all $\varphi \in C_c^\infty(U)$. We have

$$\begin{aligned} E(u+t\varphi) &= \frac{1}{2}\int_U \langle d(u+t\varphi), d(u+t\varphi)\rangle dV \\ &= E(u) + t\int_U \langle du, d\varphi\rangle dV + t^2 E(\varphi). \end{aligned}$$

Since $I_\varphi(t) := E(u+t\varphi)$ is a smooth function of t for fixed φ, and since $I_\varphi(0) \le I_\varphi(t)$ for $|t|$-small, we must have $I'_\varphi(0) = 0$. This shows that if u is a minimizer, then

$$\int_U \langle du, d\varphi\rangle dV = 0$$

for any choice of $\varphi \in C_c^\infty(U)$. By the properties of the codifferential δ, this implies that

$$\int_U (\delta du)\varphi dV = 0$$

for all $\varphi \in C_c^\infty(U)$. Thus any minimizer u has to satisfy the equation

$$\delta du = 0 \quad \text{ in } U.$$

We have arrived at the definition of the Laplace-Beltrami operator.

Definition 2.39 (Laplace-Beltrami operator)**.** The Laplace-Beltrami operator on (U, g) is defined by

$$\Delta_g u := -\delta du.$$

Lemma 2.40. The Laplace-Beltrami operator has the expression

$$\Delta_g u = |g|^{-1/2}\partial_j\big(|g|^{1/2}g^{jk}\partial_k u\big),$$

where, as before, $|g| = \det(g_{jk})$ is the determinant of g.

Proof. Exercise. □

Remark 2.41. There are differing sign conventions for the Laplace-Beltrami operator. Honoring the title of this course, we have chosen the convention which is perhaps most common in analysis and makes the Laplace-Beltrami operator for Euclidean metric equal to $\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. However, it is very common in geometry define the Laplace-Beltrami operator with the opposite sign, which has the benefit that the operator becomes positive. Moreover, in probability theory a factor of $\frac{1}{2}$ is often included in the definition. In this course we will stick to the analysts' convention so that $\Delta_g = -\delta d$.

The existence of a canonical Laplace operator associated to a Riemannian metric implies that one has analogues of the classical linear PDE:

- $\Delta_g u = 0$ (Laplace)
- $\partial_t u - \Delta_g u = 0$ (heat)
- $\partial_t^2 u - \Delta_g u = 0$ (wave)
- $i\partial_t u + \Delta_g u = 0$ (Schrödinger)

Therefore in physical terms, any Riemannian manifold will support a theory for electrostatics, heat flow, acoustic wave propagation, and quantum mechanics. Note also that the theory of geodesics leads to a version of classical mechanics, and there are many relations between the classical and quantum picture (i.e. between the geodesic flow and the Laplace-Beltrami operator).

3. Calulus on Riemannian manifolds

In this chapter we will discuss the calculus concepts from Chapter 2 in the more general setting of smooth or Riemannian manifolds. Thus, instead of working on open sets $U \subset \mathbb{R}^n$, we wish to perform calculus operations on spaces such as

- surfaces in $\mathbb{R}^3$ (spheres, tori, double tori, etc)
- n-dimensional, possibly complicated hypersurfaces $S \subset \mathbb{R}^{n+k}$
- groups of transformations ($GL(n), SO(n), U(n)$ etc)

Our aim is to present the material briefly, giving the definitions but omitting the proofs of their basic properties (for proofs see for instance [6, 7]).

3.1. Smooth manifolds.

We briefly recall the definition and basic theory of smooth manifolds.

Definition 3.1 (Smooth manifold)**.** A *smooth n-dimensional manifold* is a topological space M, assumed to be Hausdorff and second countable, together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha \colon U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n$, and $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha\big(U_\alpha \cap U_\beta\big) \to \varphi_\beta\big(U_\alpha \cap U_\beta\big)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Any family $\big\{(U_\alpha, \varphi_\alpha)\big\}$ as above is called an *atlas*. Any atlas gives rise to a maximal atlas, called a *smooth structure*, which is not strictly contained in any other atlas. We assume that we are always dealing with the maximal atlas. The pairs $(U_\alpha, \varphi_\alpha)$ are called *charts*, and the maps φ_α are called *local coordinate systems*. One usually writes $x = \varphi_\alpha$ and identifies points $p \in U_\alpha$ with points $x(p) = \varphi_\alpha(p) \in \tilde{U}_\alpha$ in $\mathbb{R}^n$.

Definition 3.2 (Smooth manifold with boundary)**.** A *smooth n-dimensional manifold with boundary* is a topological space M, assumed to be Hausdorff and second countable, together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha \colon U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n_+ := \big\{x \in \mathbb{R}^n : x_n \geq 0\big\}$, and $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha\big(U_\alpha \cap U_\beta\big) \to \varphi_\beta\big(U_\alpha \cap U_\beta\big)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Here, if $A \subset \mathbb{R}^n$ we say that a map $F \colon A \to \mathbb{R}^n$ is smooth if it extends to a smooth map $\tilde{A} \to \mathbb{R}^n$, where $\tilde{A}$ is an open set in $\mathbb{R}^n$ containing A.

If M is a manifold with boundary, we say that p is a boundary point if $\varphi(p) \in \partial\mathbb{R}^n_+$ for some chart φ, and an interior point if $\varphi(p) \in \operatorname{int}(\mathbb{R}^n_+)$ for some φ. We write ∂M for the set of boundary points and M^{int} for the set of interior points. Since M is not assumed to be embedded in any larger space, these definitions may differ from the usual ones in point set topology.

To clarify the relations between the definitions, note that a manifold is always a manifold with boundary (the boundary being empty), but a manifold with boundary is a manifold if and only if the boundary is empty (left as exercise). However, we will loosely refer to manifolds both with and without boundary as "manifolds".

We have the following classes of manifolds:

- A *closed manifold* is compact, connected, and has no boundary.
  - Examples: the sphere $\mathbb{S}^n$, the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$
- An *open manifold* has no boundary and no component is compact.
  - Examples: open subsets of $\mathbb{R}^n$, proper open subsets of a closed manifold
- A *compact manifold with boundary* is a manifold with boundary which is compact as a topological space.
  - Examples: the closures of bounded open sets in $\mathbb{R}^n$ with smooth boundary, the closures of open sets with smooth boundary in closed manifolds

Smooth maps. We recall the definition of smooth maps between manifolds.

Definition 3.3 (Smooth map)**.** Let $f\colon M \to N$ be a map between two manifolds. We say that f is *smooth* near a point p if $\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth for some charts (U, φ) of M and (V, ψ) of N such that $p \in U$ and $f(U) \subset V$.

We say that f is *smooth* in a set $A \subset M$ if it is smooth near any point of A. The set of all maps $f\colon M \to N$ which are smooth in A is denoted by $C^\infty(A, N)$. If $N = \mathbb{R}$, we write $C^\infty(A, N) = C^\infty(A)$.

Tangent bundle. If $U \subset \mathbb{R}^n$ is open, we defined the tangent space $T_xU = \{x\} \times \mathbb{R}^n$ to be a copy of $\mathbb{R}^n$ sitting at x. Any $v \in T_xU$ can be thought of as an infinitesimal direction where one can move from x, and there is a corresponding directional derivative

$$\partial_v\colon C^\infty(U) \to \mathbb{R}, \quad \partial_v f(x) := v \cdot \nabla f(x).$$

Then ∂_v is a linear operator satisfying $\partial_v(fg) = (\partial_v f)g + f(\partial_v g)$. Such an object is called a *derivation*. It turns out that derivations can be identified with vectors in the tangent space, and this leads to a definition of tangent spaces on abstract manifolds.

Definition 3.4 (Derivation)**.** Let $p \in M$. A *derivation* at p is a linear map $v\colon C^\infty(M) \to \mathbb{R}$ which satisfies the Leibniz rule $v(fg) = (vf)g(p) + f(p)(vg)$. The *tangent space* T_pM is the vector space consisting of all derivations at p. Its elements are called *tangent vectors*.

The tangent space T_pM is an n-dimensional vector space when $\dim(M) = n$. If x is a local coordinate system in a neighborhood U of p, we define the *coordinate vector fields* ∂_j for each $q \in U$ as the derivations

$$\partial_j|_q f := \frac{\partial}{\partial x_j}\big(f \circ x^{-1}\big)\big(x(q)\big), \quad j = 1, \cdots, n. \tag{3.1}$$

Then $\{\partial_j|_q\}$ is a basis of T_qM, and any $v \in T_qM$ may be written as $v = v^j\partial_j$.

The *tangent bundle* is the disjoint union

$$TM := \bigvee_{p \in M} T_pM.$$

The tangent bundle has the structure of a $2n$-dimensional manifold defined as follows. For any chart (U, x) of M, we represent elements of T_qM for $q \in U$ as $v = v^j(q)\partial_j|_q$, and define a map $\tilde{\varphi}\colon TU \to \mathbb{R}^{2n}$,

$$\tilde{\varphi}(q, v) = \big(x(q), v^1(q), \cdots, v^n(q)\big).$$

The charts $(TU, \tilde{\varphi})$ are called the *standard charts* of TM and they define a smooth structure on TM and they define a smooth structure on TM.

Since the tangent bundle is a smooth manifold, the following definition makes sense:

Definition 3.5 (Vector field)**.** A *vector field* on M is a smooth map $X\colon M \to TM$ such that $X(p) \in T_pM$ for each $p \in M$.

Cotangent bundle. The dual space of a vector space V is

$$V^* := \big\{u\colon V \to \mathbb{R} : \ u \text{ is linear } \big\}.$$

The dual space of T_pM is denoted by T_p^*M and is called the *cotangent space* of M at p. Let x be local coordinates in U and let ∂_j be the coordinate vector fields that span T_qM for $q \in U$. We denote by dx^j the elements of the dual basis of T_q^*M, so that any $\xi \in T_q^*M$ can be written as $\xi = \xi_j dx^j$. The dual basis is characterized by

$$dx^j(\partial_k) = \delta_{jk}.$$

The *cotangent bundle* is the disjoint union

$$T^*M = \bigvee_{p \in M} T_p^*M.$$

This becomes a $2n$-dimensional manifold by defining for any chart (U, φ) of M a chart $(T^*U, \tilde{\varphi})$ of T^*M by

$$\tilde{\varphi}(q, \xi_j dx^j) = \big(\varphi(q), \xi_1, \cdots, \xi_n\big).$$

Definition 3.6 (Differential 1-form)**.** A 1-form on M is a smooth map $\alpha\colon M \to T^*M$ such that $\alpha(p) \in T_p^*M$ for each $p \in M$.

Tensor bundles. If V is a finite dimensional vector space, the space of (covariant) k-tensors on V is

$$T^k(V) := \big\{u\colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R},\ u \text{ is linear in each variable}\big\}.$$

The *k-tensor bundle* on M is the disjoint union

$$T^kM = \bigvee_{p \in M} T^k\big(T_pM\big).$$

If x are local coordinates in U and dx^j is the basis for T_q^*M, then each $u \in T^k(T_qM)$ for $q \in U$ can be written as

$$u = u_{j_1\cdots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k}.$$

Here $\otimes$ is the *tensor product*

$$\otimes \colon T^k(V) \times T^s(V) \to T^{k+s}(V), \quad (u_0, u_1) \mapsto u_0 \otimes u_1,$$

where for $v_0 \in V^k$ and $v_1 \in V^s$, we define

$$\big(u_0 \otimes u_1\big)(v_0, v_1) := u_0(v_0)u_1(v_1).$$

It follows that the elements $dx^{j_1} \otimes \cdots \otimes dx^{j_k}$ span $T^k\big(T_qM\big)$. Similarly as above, T^kM has the structure of a smooth manifold of dimension $n + n^k$.

Definition 3.7 (Tensor field)**.** A *k-tensor field* on M is a smooth map $u \colon M \to TM$ such that $u(p) \in T^k\big(T_pM\big)$ for each $p \in M$.

Exterior powers. The space of alternating k-tensor is

$$\Lambda^k(V) := \big\{u \in T^k(V) : u(v_1, \cdots, v_k) = 0 \text{ if } v_i = v_j \text{ for some } i \neq j\big\}.$$

To describe a basis for $\Lambda^k(T_pM)$, we introduce the *wedge product*

$$\begin{aligned} \wedge \colon \Lambda^k(V) \times \Lambda^s(V) &\to \Lambda^{k+s}(V) \\ (\omega_0, \omega_1) &\mapsto \omega_0 \wedge \omega_1 := \frac{(k+s)!}{k!s!} \operatorname{Alt}\big(\omega_0 \otimes \omega_1\big), \end{aligned}$$

where $\operatorname{Alt} \colon T^k(V) \to \Lambda^k(V)$ is the projection to alternating tensors defined as follows

$$\operatorname{Alt}(T)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{Sign}(\sigma) T\big(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\big).$$

Here, we use S_k for the group of permutations σ of $\{1, \cdots, k\}$ and $\operatorname{Sign}(\sigma)$ for the signature of $\sigma \in S_k$.

The following properties of the wedge product can be checked from the definition:

Lemma 3.8 (Computation law for wedge product)**.** The wedge product is *associative*, i.e. $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$ for any alternating tensors ω_i. Moreover, if $\omega_1, \cdots, \omega_k$ are 1-tensors, then

$$\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(k)} = (-1)^{\operatorname{Sign}(\sigma)} \omega_1 \wedge \cdots \wedge \omega_k, \ \sigma \in S_k \tag{3.2}$$

and for any $v_1, \cdots, v_k \in V$ one has

$$\big(\omega_1 \wedge \cdots \wedge \omega_k\big)(v_1, \cdots, v_k) = \det \begin{bmatrix} \omega_1(v_1) & \cdots & \omega_1(v_k) \\ \vdots & \ddots & \vdots \\ \omega_k(v_1) & \cdots & \omega_k(v_k) \end{bmatrix}. \tag{3.3}$$

Proof. Exercise. □

If x is a local coordinate system in U, then a basis of $\Lambda^k(T_pM)$ is given by

$$\big\{dx^{j_1} \wedge \cdots \wedge dx^{j_k}\big\}_{1 \le j_1 < j_2 < \cdots < j_k \le n}.$$

One can prove that $\Lambda^k(M)$ is a smooth manifold of dimension $n + \binom{n}{k}$.

Definition 3.9 (Differential k-form). A *k-form* on M is a smooth map $\omega\colon M \to TM$ such that $\omega(p) \in \Lambda^k(T_pM)$ for each $p \in M$.

Smooth sections. The above constructions of the tangent bundle, cotangent bundle, tensor bundles, and exterior powers are all examples of *vector bundles* with base manifold M. We will not need a precise definition here, but just note that in each case there is a natural vector space over any point $p \in M$ (called the *fiber over* p).

Definition 3.10 (Smooth section). A smooth section of a vector bundle E over M is a smooth map $s\colon M \to E$ such that for each $p \in M$, $s(p)$ belongs to the fiber over p. The space of smooth sections of E is denoted by $C^\infty(M, E)$.

We have the following terminology:

- $C^\infty(M, TM)$ is the set of *vector fields* on M;
- $C^\infty(M, T^kM)$ is the set of *k-tensor fields* on M;
- $\Omega^1(M) = C^\infty(M, T^*M)$ is the set of differential 1-*forms* on M;
- $\Omega^k(M) = C^\infty(M, \Lambda^k(M))$ is the set of differential *k-forms* on M.

Let x be local coordinates in a set U, and let ∂_j and dx^j be the coordinate vector fields and 1-forms in U, which span T_qM and T_q^*M, respectively, for $q \in U$. In these local coordinates,

- a vector field X has the expression $X = X^j\partial_j$
- a 1-form α has the expression $\alpha = \alpha_j dx^j$
- a k-tensor field u can be written as
$$u = u_{j_1\cdots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$
- a k-form ω has the form
$$\omega = \omega_I dx^I,$$
where $I = (i_1, \cdots, i_k)$ and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, with the sum being over all I such that $1 \le i_1 < i_2 < \cdots < i_k \le n$

Here, the component functions $X^j, \alpha_j, u_{j_1\cdots j_k}$ and ω_I are all smooth real valued functions in U.

We mention briefly how the local coordinate formula for a k-tensor field u is obtained. If (U, x) is a local coordinate chart and $\{\partial_j\}$ are the associated coordinate vector fields, one can write any $v \in T_qM$ for $q \in U$ as $v = v^k\partial_k|_q$ for some $(v^1, \cdots, v^n) \in \mathbb{R}^n$. Thus by linearity

$$u_q(v_1, \cdots, v_k) = u_q(v_1^{j_1}\partial_{j_1}|_q, \cdots, v_k^{j_k}\partial_{j_k}|_q) = u_q(\partial_{j_1}|_q, \cdots, \partial_{j_k}|_q)v_1^{j_1}\cdots v_k^{j_k}.$$

If we define

$$u_{j_1\cdots j_k}(q) := u_q(\partial_{j_1}|_q, \cdots, \partial_{j_k}|_q),$$

then the above computation and the definition of tensor product imply

$$u_q(v_1, \cdots, v_k) = \big(u_{j_1\cdots j_k}(q) dx^{j_1}|_q \otimes \cdots \otimes dx^{j_k}|_q\big)(v_1, \cdots, v_k).$$

This proves that the local coordinate representation of a tensor field u is obtained just by evaluating u at coordinate vector fields.

Example 3.11. Some examples of the smooth sections that will be encountered in this text are:

- Vector fields: the gradient vector field $\mathrm{grad}(f)$ for $f \in C^\infty(M)$, coordinate vector fields ∂_j in a chart U
- One-forms: the exterior derivative df for $f \in C^\infty(M)$
- 2-tensor fields: Riemannian metrics g, Hessians $\mathrm{Hess}(f)$ for $f \in C^\infty(M)$, Ricci curvature R_{ab}
- 4-tensor fields: Riemann curvature tensor R_{abcd}
- n-forms: the volume form dV in an n-dimensional Riemannian manifold (M, g)

Change of coordinates. We next consider the transformation laws for vector and tensor fields under changes of coordinates. It is convenient to phrase these in terms of more general pullbacks or pushforwards by smooth maps between manifolds. We begin with pushforwards of tangent vectors.

Definition 3.12 (Push-forward)**.** Let $F\colon M \to N$ be a smooth map. The *push-forward* by F is the map acting on T_pM for any $p \in M$ by

$$F_*\colon T_pM \to T_{F(p)}N,\ F_*v(f) = v\big(f \circ F\big) \text{ for } f \in C^\infty(N).$$

The map F_* is also called the *derivative* or *tangent map* of F, and we sometimes denote it by DF.

We now compute how F_* transforms vector fields in local coordinates.

Lemma 3.13. Let $F\colon M \to N$ be a smooth map and let X be a vector field in M. If (U, y) and (V, z) are coordinate charts near p in M and near $F(p)$ in N, respectively, and if Y and Z are corresponding coordinate representative of X and F_*X so that

$$X(q) = Y^j\big(y(q)\big)\partial_{y^j}|_q, \quad F_*X(r) = Z^k\big(z(r)\big)\partial_{z^k}|_r,$$

then

$$Z^k\big(z(F(q))\big) = \partial_{y^j}\tilde{F}^k\big(y(q)\big)Y^j\big(y(q)\big),$$

where $\tilde{F} = z \circ F \circ y^{-1}$.

Proof. Given $q \in U$ with $F(q) \in V$, the tangent vector $F_*X|_{F(q)}$ is a derivation acting on $f \in C^\infty(N)$ and we have

$$\begin{aligned}
F_*X|_{F(q)}f &= X|_q(f \circ F) = Y^j(y(q))\partial_{y^j}|_q\big(f \circ z^{-1} \circ \tilde{F} \circ y\big) \\
&= Y^j(y(q))\partial_{y^j}\big((f \circ z^{-1}) \circ \tilde{F}\big)(y(q)) \\
&= Y^j(y(q))\partial_{z^k}(f \circ z^{-1})\big(z(F(q))\big)\partial_{y^j}\tilde{F}^k(y(q)) \\
&= \partial_{y^j}\tilde{F}^k(y(q))Y^j(y(q))\partial_{z^k}|_{F(q)}f.
\end{aligned}$$

□

Remark 3.14. Applying Lemma 3.13 to the inclusion map $F = i\colon M \to N$ shows that the representatives Y and Z of a vector field X in two coordinate charts (U, y) and (V, z) with $U \cap V \neq \emptyset$ are related by

$$Z^k\big(z(q)\big) = \partial_{y^j}\big(z \circ y^{-1}\big)^k\big(y(q)\big)Y^j\big(y(q)\big), \quad q \in U \cap V. \tag{3.4}$$

This provides an alternative way to define vector fields on a manifold: if to each coordinate chart (U, y) on M one associates a vector field Y in $y(U) \subset \mathbb{R}^n$, and if the vector fields Y and Z for any two coordinate charts (U, y) and (V, z) with $U \cap V \neq \emptyset$ satisfying (3.4), then there is a vector field X in M whose coordinate representation in any chart (U, y) is Y. If (3.4) holds, we say that the coordinate representations Y *transform as a vector field in M*.

We now move to tensor fields. If $F\colon M \to N$ is a smooth map, we can associate to a tensor field $u \in C^\infty(N, T^kN)$ a corresponding tensor field $F^*u \in C^\infty(M, T^kM)$ in the following way.

Definition 3.15 (Pullback)**.** Let $F\colon M \to N$ be a smooth map. The *pullback* by F acting on k-tensor fields is the map $F^*\colon C^\infty(N, T^kN) \to C^\infty(M, T^kM)$ given by

$$\big(F^*u\big)_p(v_1, \cdots, v_k) = u_{F(p)}(F_*v_1, \cdots, F_*v_k) \quad \text{for } v_1, \cdots, v_k \in T_pM.$$

It is easy to see that F^*u is indeed a tensor field on M and that F^* has the following properties.

Lemma 3.16 (Basic properties of F^*)**.** Let $F\colon M \to N$ be a smooth map, let $f \in C^\infty(N)$, let u_0 and u_1 be tensor fields in N, and let ω_0 and ω_1 be differential forms in N. Then

- $F^*(fu_0) = (f \circ F)F^*u_0$
- $F^*(u_0 \otimes u_1) = F^*u_0 \otimes F^*u_1$
- F^* preserves alternating tensors and thus induces a map on differential forms,

$$F^*\colon \Omega^k(N) \to \Omega^k(M), \quad 0 \leq k \leq n$$

- $F^*(\omega_0 \wedge \omega_1) = F^*\omega_0 \wedge F^*\omega_1$.

Proof. Left as an exercise. □

In terms of local coordinates, the pullback acts by

- $F^*f = f \circ F$ if f is a smooth function (0-form)
- $F^*(\alpha_j dx^j) = (\alpha_j \circ F)d(x^j \circ F) = (\alpha_j \circ F)dF^j$ if α is a 1-form and it has the following expression for higher order tensors:

Lemma 3.17. Let $F\colon M \to N$ be a smooth map and let u be a k-tensor field in N. If (U, y) and (V, z) are coordinate charts near p in M and near $F(p)$ in N, respectively, and if $(y_{i_1\cdots i_k})$ and $(z_{j_1\cdots j_k})$ are corresponding coordinate representations of F^*u and u so that

$$F^*u(q) = y_{i_1\cdots i_k}\big(y(q)\big)dy^{i_1} \otimes \cdots \otimes dy^{i_k}|_q,$$
$$u(r) = z_{j_1\cdots j_k}\big(z(r)\big)dz^{j_1} \otimes \cdots \otimes dz^{j_k}|_r,$$

then

$$y_{i_1\cdots i_k}|_{y(q)} = \big(\partial_{y^{i_1}}\tilde{F}^{j_1}\big)\cdots\big(\partial_{y^{i_k}}\tilde{F}^{j_k}\big)|_{y(q)},$$

where $\tilde{F} = z \circ F \circ y^{-1}$.

Proof. Given $q \in U$ with $F(q) \in V$, we have

$$\begin{aligned} y_{i_1\cdots i_k}\big(y(q)\big) &= F^*u|_q(\partial_{y^{i_1}}, \cdots, \partial_{y^{i_k}}) = u|_{F(q)}(F_*\partial_{y^{i_1}}, \cdots, F_*\partial_{y^{i_1}}) \\ &\overset{\text{Lemma 3.13}}{=} u|_{F(q)}\big(\partial_{y^{i_1}}\tilde{F}^{j_1}(y(q))\partial_{z^{j_1}}, \cdots, \partial_{y^{i_k}}\tilde{F}^{j_k}(y(q))\partial_{z^{j_k}}\big) \\ &= \partial_{y^{i_1}}\tilde{F}^{j_1}(y(q))\cdots\partial_{y^{i_k}}\tilde{F}^{j_k}(y(q))z_{j_1\cdots j_k}(z(F(q))). \end{aligned}$$

□

Remark 3.18. We have defined F_* acting on vector fields and F^* acting on k-tensor fields. If $F\colon M \to N$ is a diffeomorphism, one can define in general $F_* = (F^{-1})^*$ and $F^* = (F^{-1})_*$, and thus for a diffeomorphism F the pushforward and pullback are defined both on vector and tensor fields.

Exterior derivative. The exterior derivative d is a first order differential operator mapping differential k-forms to $k+1$-forms. It can be defined first on 0-forms (that is, smooth functions f) by the local coordinate expression

$$df := \frac{\partial f}{\partial x_j}dx^j.$$

In general, if $\omega = \omega_I dx^I$ is a k-form, we define

$$d\omega := d\omega_I \wedge dx^I.$$

Lemma 3.19. *The definition of d is independent of the choice of local coordinates, and $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear map for $0 \le k \le n$. The operator d has the properties*

- $d^2 = 0$
- $d|_{\Omega^n(M)} = 0$
- $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$ *for a k-form ω and s-form θ*
- $F^*(d\omega) = dF^*\omega$.

Proof. Left as an exercise. □

Partition of unity. A major reason for including the condition of second countability in the definition of manifolds is to ensure the existence of *partitions of unity.* These make it possible to make constructions in local coordinates and then glue them together to obtain a global construction.

Lemma 3.20 (Partition of unity)**.** *Let M be a manifold and let $\{U_\alpha\}$ be an open cover. There exists a family of C^∞ functions $\{\chi_\alpha\}$ on M such that $0 \le \chi_\alpha \le 1$, $\mathrm{supp}(\chi_\alpha) \subset U_\alpha$, any point of M has a neighborhood which intersects only finitely many of the sets $\mathrm{supp}(\chi_\alpha)$, and furhter*

$$\sum_\alpha \chi_\alpha = 1 \quad \text{in } M.$$

The partition $\{\chi_\alpha\}$ as in Lemma 3.20 is called *a smooth partition of unity subordinate to the open cover* $\{U_\alpha\}$.

Integration on manifolds. The natural objects that can be integrated on an n-dimensional manifold are the differential n-forms. This is due to the transformation law for n-forms in $\mathbb{R}^n$ under smooth diffeomorphisms F in $\mathbb{R}^n$,

$$\begin{aligned} F^*(dx^1 \wedge \cdots \wedge dx^n) &= dF^1 \wedge \cdots \wedge dF^n \\ &= (\partial_{j_1} F^1) \cdots (\partial_{j_n} F^n) dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\ &= (\det DF) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

This is almost the same as the transformation law for integrals in $\mathbb{R}^n$ under changes of variables, the only difference being that in the latter the factor $|\det DF|$ instead $\det DF$ appears. To define an invariant integral, we therefore need to make sure that all changes of coordinates have positive Jacobian.

Definition 3.21 (Orientation)**.** If M admits a smooth nonvanishing n-form, we say that M is *orientable*. An *oriented manifold* is a manifold together with a given nonvanishing n-form.

If M is oriented with a given n-form Ω, a basis $\{v_1, \cdots, v_n\}$ of T_pM is called *positive* if $\Omega(v_1, \cdots, v_n) > 0$. There are many n-forms on an oriented manifold which give the same positive bases; we call any such n-form an *orientation form.* If (U, φ) is a connected coordinate chart, we say that this chart is *positive* if the coordinate vector fields $\{\partial_1, \cdots, \partial_n\}$ form a positive basis of T_qM for all $q \in M$.

A map $F\colon M \to N$ between two oriented manifolds is said to be *orientation preserving* if it pulls back an orientation form on N to an orientation form of M. In terms of local coordinates given by positive charts, one can see that a map is orientation preserving if and only if its Jacobian determinant is positive.

Example 3.22. The standard orientation of $\mathbb{R}^n$ is given by the n-form $dx^1 \wedge \cdots \wedge dx^n$, where x are the Cartesian coordinates.

If ω is a compactly supported n-form in $\mathbb{R}^n$, we may write $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some smooth compactly supported function f. Then the integral of ω is defined by

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f(x) dx^1 \cdots dx^n.$$

If ω is a smooth n-form in a manifold M whose support is compactly contained in U for some positive chart (U, φ), then the integral of ω over M is defined by

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

Finally, if ω is a compactly supported n-form in a manifold M, the integral of ω over M is defined by

$$\int_M \omega := \sum_j \int_{U_j} \chi_j \omega,$$

where $\{U_j\}$ is some open cover of $\operatorname{supp}(\omega)$ by positive charts, and $\{\chi_j\}$ is a partition of unity subordinate to the cover $\{U_j\}$.

It is easy to see that the above definition of integral is independent of the choice of positive charts and the partition of unity (Exercise).

The following result is a basic integration by parts formula which implies the usual theorems of Gauss and Green.

Theorem 3.23 (Stokes theorem)**.** If M is an oriented manifold with boundary and if ω is a compactly supported $(n-1)$-form on M, then

$$\int_M d\omega = \int_{\partial M} i^*\omega,$$

where $i\colon \partial M \to M$ is the natural inclusion.

Here, if M is an oriented manifold with boundary, then ∂M has a natural orientation defined as follows: for any point $p \in \partial M$, a basis $\{E_1, \cdots, E_{n-1}\}$ of $T_p(\partial M)$ is defined to be positive if $\{N_p, E_1, \cdots, E_{n-1}\}$ is a positive basis of T_pM where N is some outward pointing vector field near ∂M (that is, there is a smooth curve $\gamma\colon [0,\varepsilon) \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = -N_p$). One can show that any manifold with boundary has an outward pointing vector field, and that the above definition gives a valid orientation on ∂M.

3.2. **Riemannian manifolds. Riemannian metrics.** If u is a 2-tensor on M, we say that u is *symmetric* if $u(v,w) = u(w,v)$ for any tangent vectors v, w and that u is *positive definite* if $u(v,v) > 0$ unless $v = 0$.

Definition 3.24 (Riemannian metric)**.** A *Riemannian metric* on a manifold M is a symmetric positive definite 2-tensor field g on M. The pair (M,g) is called a *Riemannian manifold.*

If g is a Riemannian metric on M, then $g_p\colon T_pM \times T_pM \to \mathbb{R}$ is an inner product on T_pM for any $p \in M$. As before, we shall write

$$\langle v, w\rangle := g(v,w), \quad |v| := \langle v, v\rangle^{\frac{1}{2}}.$$

In local coordinates, a Riemannian metric is just a positive definite symmetric matrix. To see this, let (U,x) be a chart on M, and write $v, w \in T_qM$ for $q \in U$ in terms of the coordinate vector fields ∂_j as $v = v^j\partial_j$ and $w = w^k\partial_k$. Then

$$g(v,w) = g(\partial_j, \partial_k)v^jw^k.$$

This shows that g has the local coordinate expression

$$g = g_{jk}dx^j \otimes dx^k,$$

where $g_{jk} := g(\partial_j, \partial_k)$ and the matrix $(g_{jk})_{j,k=1}^n$ is symmetric and positive definite. We will also write $(g^{jk})_{j,k=1}^n$ for the inverse matrix of (g_{jk}) and $|g| = \det(g_{jk})$ for the determinant.

Example 3.25. Some examples of Riemannian manifolds:

(1) *Euclidean space.* If U is a bounded open set in $\mathbb{R}^n$, then (U, g_0) is a Riemannian manifold, where g_0 is the *Euclidean metric* for which $g_0(v, w) = v \cdot w$ is the Euclidean inner product of $v, w \in T_pU \approx \mathbb{R}^n$. In Cartesian coordinates, g_0 is just the identity matrix.
(2) If U is as above, then more generally (U, g) is a Riemannian manifold if $g(x) = \big(g_{jk}(x)\big)_{j,k=1}^n$ is any family of positive definite symmetric matrices whose elements depend smoothly on $x \in U$.
(3) If U is a bounded open set in $\mathbb{R}^n$ with smooth boundary, then $(\bar{U}, g)$ is a compact Riemannian manifold with boundary if $g(x)$ is a family of positive definite symmetric matrices depending smoothly on $x \in \bar{U}$.
(4) *Hypersurfaces.* Let S be a smooth hypersurface in $\mathbb{R}^n$ such that $S = f^{-1}(0)$ for some smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ which satisfies $\nabla f \neq 0$ when $f = 0$. Then S is a smooth manifold of dimension $n - 1$, and the tangent space T_pS for any $p \in S$ can be identified with $\{v \in \mathbb{R}^n : v \cdot \nabla f(p) = 0\}$. Using this identification, we define an inner product $g_p(v, w)$ on T_pS by taking the Euclidean inner product of v and w interpreted as vectors in $\mathbb{R}^n$. Then (S, g) is a Riemannian manifold, and g is called the *induced Riemannian metric* on S (this metric being induced by the Euclidean metric in $\mathbb{R}^n$).
(5) *Immersed manifold.* Let $f\colon M \to N$ be an *immersion*, i.e. f is smooth with $df_p\colon T_pM \to T_{f(p)}N$ being injective for all $p \in M$. If N has a Riemannian structure h, then f *induces a Riemannian structure* g on M by defining for each $p \in M$,
$$g_p(v, w) := h_{f(p)}\big(f_*(v), f_*(w)\big) = h_{f(p)}\big(df_p(v), df_p(w)\big)$$
for all $v, w \in T_pM$. Since df_p is injective, $g_p(\cdot, \cdot)$ is positive definite. It is easy to see that g is a symmetric positive definite 2-tensor field and this Riemannian metric is called the metric *induced* by f, and f is an *isometric immersion.*
(6) *Model spaces.* The model spaces of Riemannian geometry are the Euclidean space $(\mathbb{R}^n, g_0)$, the sphere $(\mathbb{S}^n, g)$, where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$ and g is the induced Riemannian metric, and the hyperbolic space $(\mathbb{H}^n, g)$, which may be realized by taking $\mathbb{H}^n$ to be the unit ball in $\mathbb{R}^n$ with metric $g_{jk}(x) = \frac{4}{(1-|x|^2)^2}\delta_{jk}$.

The Riemannian metric allows to measure lengths and angles of tangent vectors on a manifold, the *length* of a vector $v \in T_pM$ being $|v|$ and the *angle* between two vectors $v, w \in T_pM$ being the number $\theta(v, w) \in [0, \pi]$ which satisfies

$$\cos\theta(v, w) := \frac{\langle v, w\rangle}{|v||w|}. \tag{3.5}$$

Physically, one may think of a Riemannian metric g as the resistivity of a conducting medium (the conductivity matrix (γ^{jk}) of the medium corresponds formally to $(|g|^{\frac{1}{2}} g^{jk})$), or as the inverse of sound speed squared in a medium where acoustic waves propagate (if a medium $U \subset \mathbb{R}^n$ has scalar sound speed $c(x)$, then a natural Riemannian metric is $g_{jk}(x) = c(x)^{-2}\delta_{jk}$). In the latter case, regions where g is large (resp. small) correspond to low velocity regions (resp. high velocity regions). We will later define geodesics, which are length minimizing curves on a Riemannian manifold, and these tend to avoid low velocity regions as one would expect.

The following result ensures the existence of Riemannian metric on a given smooth manifold.

Theorem 3.26 (Existence of Riemannian metric)**.** Every smooth manifold admits a Riemannian metric.

Proof. Let $\{U_\alpha\}$ be an open covering of M. By Lemma 3.20, we may assume that $\{U_\alpha\}$ is a locally finite open covering of M, i.e. each point $p \in M$ has a neighborhood V_p such that $V_p \cap U_\alpha \neq \emptyset$ for at most finitely many α and we may find $\{\chi_\alpha\}$ subordinate to $\{U_\alpha\}$ such that χ_α satisfies the conditions in Lemma 3.20.

It is clear that we can define a Riemannian metric g^α on each U_α: simply take the metric induced by the local coordinate. Then we glue all the g^α to form a Riemannian metric g by

$$g(v,w) := \sum_\alpha \chi_\alpha(p) g_p^\alpha(v,w) \qquad \text{for all } p \in M \text{ and } v, w \in T_pM.$$

It is easy to verify that this construction defines a Riemannian metric on M. □

As in Section 2.5, a smooth curve $\gamma\colon [a,b] \to M$ is said to be *regular* if $\gamma'(t) \neq 0$ for all $t \in [a,b]$. For a regular curve $\gamma\colon [a,b] \to M$, we define its *length* as

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt.$$

The length of a piecewise regular curve is defined as the sum of lengths of the regular parts. For each pair of points $p, q \in M$, the *Riemannian distance* $d_g(p,q)$ between p and q is defined to be the infimum of the lengths of all piecewise regular curves joining p and q.

The following basic result implies that it makes sense to apply all the concepts of the theory of metric space to a connected Riemannian manifold (M,g).

Theorem 3.27 (Riemannian manifold as metric spaces)**.** Let (M,g) be a connected Riemannian manifold (with or without boundary). Then (M,d_g) is metric space whose metric topology is the same as the given manifold topology.

The proof of Theorem 3.27 requires the following technical result.

Lemma 3.28. Let (M,g) be a Riemannian manifold (with or without boundary) and let d_g be its Riemannian distance function. Suppose U is an open subset of M and $p \in U$. Then p has a coordinate neighborhood $V \subset U$ with the property that there are positive constants C_0, C_1 satisfying the following inequalities:

i). If $q \in V$, then $d_g(p,q) \leq C_0 d_{g_0}(p,q)$, where g_0 is the standard Euclidean metric on V;
ii). If $q \in M \backslash V$, then $d_g(p,q) \geq C_1$.

Proof. Exercise. □

Proof of Theorem 3.27. It is immediate from the definition of d_g that $d_g(p,q) = d_g(q,p) \geq 0$ and $d_g(p,p) = 0$. On the other hand, suppose $p, q \in M$ are distinct. Let $U \subset M$ be an open set that contains p but not q, and choose a coordinate neighborhood V of p contained in U and satisfying the conclusion of Lemma 3.28. Then Lemma 3.28 ii) shows that $d_g(p,q) \geq C_1 > 0$.

The triangle inequality follows from the fact that an admissible curve from p to q can be combined with one from q to r (possibly changing the starting time of the parametrization of the second) to yield one from p to r whose length is the sum of the lengths of the two given curves (This is one reason for defining distance using piecewise regular curves instead of just regular ones). This completes the proof that d_g turns M into a metric space.

It remains to show that the metric topology is the same as the manifold topology. Suppose first that $U \subset M$ is open in the manifold topology. For each $p \in U$, we can choose a coordinate neighborhood V of p contained in U with positive constants C_0, C_1 satisfying the conclusions of Lemma 3.28. The contrapositive of part ii) of Lemma 3.28 says $d_g(p,q) < C_1 \Rightarrow q \in V \subset U$, which means that the metric ball of radius C_1 is contained in U. Thus U is open in the metric topology induced by d_g.

On the other hand, suppose U' is open in the metric topology. Given $p \in U'$, choose $\delta > 0$ such that the d_g-metric ball of radius δ around p is contained in U'. Let V be any neighborhood of p that is open in the manifold topology and satisfies the conclusions of Lemma 3.28, with corresponding constants $C_0.C_1$ (We are not claiming that $V \subset U'$). Choose ε small enough that $C_0\varepsilon < \delta$. Lemma 3.28 i) shows that if q is a point of V such that $d_{g_0}(p,q) < \varepsilon$, then $d_g(p,q) \leq C_0\varepsilon < \delta$ and thus q lies in the metric ball of radius δ about p, and hence in U'. Since the set $\{q \in V : d_{g_0}(p,q) < \varepsilon\}$ open in the given manifold topology, this shows that U' is also open in the manifold topology. □

Thanks to the preceding theorem, it makes sense to apply all the concepts of the theory of *metric spaces* to a *connected Riemannian manifold* (M, g). For example, we say that M is *(metrically) complete* if every Cauchy sequence in M converges. A subset $A \subset M$ is *bounded* if there is a positive constant C such that $d_g(p,q) \leq C$ for all $p, q \in A$; if this is the case, the *diameter of* A is the smallest such constant:

$$\operatorname{diam}(A) := \sup\{d_g(p,q) : p, q \in A\}.$$

Since every compact metric space is bounded, every compact connected Riemannian manifold (with or without boundary) has finite diameter.

Isometries. Let (M, g) and (N, h) be two Riemannian manifolds. We say that a map F is a *Riemannian isometry* from (M, g) to (N, h) if $F \colon M \to N$ is a diffeomorphism and $F^*h = g$, or more precisely,

$$g_p(v, w) = h_{F(p)}(F_*v, F_*w), \quad v, w \in T_pM.$$

Being Riemannian isometric is an equivalence relation in the class of Riemannian manifolds, and one thinks of Riemannian isometric manifolds as being identical in terms of their Riemannian structure.

Proposition 3.29 (Isometry Invariance of the Riemannian Distance Function)**.** Suppose (M, g) and (N, h) are connected Riemannian manifolds with or withoutvboundary, and $F\colon M \to N$ is a Riemannian isometry. Then

$$d_h(F(x), F(y)) = d_g(x, y)$$

for all $x, y \in M$.

Proof. This is immediate from the definition. □

Let (X, d_X) and (Y, d_Y) be two metric spaces. A homeomorphism $F\colon (X, d_X) \to (Y, d_Y)$ is called a *distance-preserving homeomorphism* (or *isometry*) if

$$d_Y(F(x), F(x')) = d_X(x, x')$$

for all $x, x' \in X$. The previous theorem says that a Riemannian isometry is always a distance-preserving homeomorphism. The converse is actually also true:

Theorem 3.30 (Mayers and Steeurid)**.** Suppose (M, g) and (N, h) are connected Riemannian manifolds with or without boundary, and $F\colon M \to N$ is a distance-preserving homeomorphism. Then F is a Riemannian isometry.

Raising and lowering indices. On a Riemannian manifold (M, g), there is a canonical way of converting tangent vectors into cotangent vectors and vice versa. We define a map

$$\flat\colon T_pM \to T_p^*M, \quad v \mapsto v^\flat$$

by requiring that $v^\flat(w) = \langle v, w\rangle$. This map, called the *flat* operator, is an isomorphism, which is given in local coordinate by

$$(v^j\partial_j)^\flat = v_j dx^j, \quad \text{where } v_j := g_{jk}v^k.$$

We say that $v^\flat$ is the cotangent vector obtained from v by *lowering indices* with respect to the metric g. The inverse of this map is the *sharp* operator

$$\sharp\colon T_p^*M \to T_pM, \quad \xi \mapsto \xi^\sharp$$

given in local coordinate by

$$(\xi_j dx^j)^\sharp = \xi^j\partial_j, \quad \text{where } \xi^j := g^{jk}\xi_k.$$

We say that $\xi^\sharp$ is obtained from ξ by *raising indices* with respect to the metric g.

Innear product of tensors. If (M, g) is a Riemannian manifold, we can use the Riemannian metric g to define inner products of tensors in a canonical way. The inner product of cotangent vectors is defined via the sharp operator by

$$\langle \alpha, \beta\rangle := \langle \alpha^\sharp, \beta^\sharp\rangle.$$

In local coordinates, one has $\langle \alpha, \beta\rangle = g^{jk}\alpha_j\beta_k$ and $g^{jk} = \langle dx^j, dx^k\rangle$.

More generally, if u and v are k-tensors with local coordinate representations $u = u_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$, we can define

$$\langle u, v\rangle := g^{i_1j_1}\cdots g^{i_kj_k}u_{i_1\cdots i_k}v_{j_1\cdots j_k}. \tag{3.6}$$

This definition turns out to be independent of the choice of coordinates, and it gives a valid inner product on k-tensors. This inner product is natural in the sense that for any diffeomorphism F onto M,

$$F^*\big(\langle u,v\rangle_g\big) = \langle F^*u, F^*v\rangle_{F^*g}.$$

Orthonormal frames. If U is an open subset of M, we say that a set $\{E_1,\cdots,E_n\}$ of vector fields in U is a *local orthonormal frame* if $\{E_1(q),\cdots,E_n(q)\}$ forms an orthonormal basis of T_qM for any $q\in U$.

Lemma 3.31 (Existence of local orthonormal frame)**.** If (M,g) is a Riemannian manifold, then for any point $p\in M$, there is a local orthonormal frame in some neighborhood of p.

Proof. Left as exercise. Applying the Gram-Schmidt orthonormalization procedure to a basis $\{\partial_j\}$ of coordinate vector fields. □

If $\{E_j\}$ is a local orthonormal frame, then the dual frame $\{\varepsilon^j\}$, which is characterized by $\varepsilon^j(E_k)=\delta_{jk}$ gives an orthonormal basis of T_q^*M for any q near p. The inner product in (3.6) is the unique inner product on k-tensor fields such that $\{\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_k}\}$ gives an orthonormal basis of $T^k\big(T_qM\big)$ for q near p whenever $\{\varepsilon^j\}$ is a local orthonormal frame of 1-forms near p.

If $\{E_j\}_{j=1}^n$ is any smooth local frame for TM on an open subset $U\subset M$ and $\{\varepsilon^i\}_{i=1}^n$ is its dual coframe, we can write g locally in U as

$$g = g_{ij}\varepsilon^i\varepsilon^j,$$

where $g_{ij}(p)=g_p(E_i|_p,E_j|_p)$, the matrix-valued function (g_{ij}) is symmetric and smooth. If $\{E_j\}$ is orthonormal, then $g_p(E_i|_p,E_j|_p)=\delta_{ij}$, in which case g has the local expression

$$g = (\varepsilon^1)^2+\cdots+(\varepsilon^n)^2,$$

where $(\varepsilon^i)^2$ denotes the symmetric product $\varepsilon^i\varepsilon^i=\varepsilon^i\otimes\varepsilon^i$.

Volume form, integration, and Sobolev spaces. From this point on, *all Riemannian manifolds will be assumed to be oriented* in order for the volume form to be defined. Clearly near any point p in (M,g), there is a positive local orthonormal frame (that is, a local orthonormal frame $\{E_j\}$ which gives a positive orthonormal basis of T_qM for q near p).

Lemma 3.32 (Existence of volume form)**.** Let (M,g) be a Riemannian n-manifold. There is a unique n-form on M, denoted by dV_g and called the volume form, such that $dV_g(E_1,\cdots,E_n)=1$ for any positive local orthonormal frame $\{E_j\}$. In local coordinates

$$dV_g = |g|^{\frac{1}{2}}dx^1\wedge\cdots\wedge dx^n.$$

The volume form is natural in the sense that $F^*(dV_g)=dV_{F^*g}$ for any orientation preserving diffeomorphism F.

Proof. Exercise. □

If f is a function on (M,g), we can use the volume form to obtain an n-form fdV_g. The integral of f over M is then defined to be the integral of the n-form fdV_g. Thus, on a Riemannian n-manifold there is a canonical way to integrate functions (instead of just n-forms).

Codifferential. Using the inner product on k-forms, we can define the codifferential operator δ as the adjoint of the exterior derivative via the relation

$$(\delta u, v) = (u, dv),$$

where $u \in C^\infty(M, \Lambda^k)$ and $v \in C_c^\infty(\mathrm{int}(M), \Lambda^{k-1})$. Applying Theorem 2.38 in coordinate neighborhood covering M and using a partition of unity, we obtain the following

Theorem 3.33 (Codifferential)**.** Let (M, g) be a Riemannian n-manifold. For each $k \in \mathbb{N}$ with $0 \le k \le n$, there is a unique linear operator

$$\delta \colon \Omega^k(M) \to \Omega^{k-1}(M)$$

having the property

$$(du, v)_{L^2} = (u, \delta v)_{L^2}, \quad u \in \Omega_c^{k-1}(M),\ v \in \Omega^k(M). \tag{3.7}$$

The operator δ satisfies $\delta \circ \delta = 0$ and $\delta|_{\Omega^0(M)} = 0$. In any local coordinates (U, x), it is a linear first order differential operator acting on component functions, and on 1-form $\beta = \beta_j dx^j$, it is given by

$$\delta\beta := -|g|^{-\frac{1}{2}}\partial_j\big(|g|^{\frac{1}{2}} g^{jk}\beta_k\big), \quad \beta = \beta_k dx^k \in \Omega^1(U). \tag{3.8}$$

It follows that $\delta\alpha$ is related to the divergence of vector fields by $\delta\alpha = -\operatorname{div}(\alpha^\sharp)$, where the divergence is defined in local coordinates by

$$\operatorname{div}(X) := |g|^{-\frac{1}{2}}\partial_j\big(|g|^{\frac{1}{2}} X^j\big).$$

Laplace-Beltrami operator. On any Riemannian manifold there is a canonical second order elliptic operator, called the *Laplace-Beltrami* operator, which is an analogue of the usual Laplacian in $\mathbb{R}^n$. As in Section 2.7, we can start from the Dirichlet energy functional

$$E(v) = \frac{1}{2}\int_M |dv|^2 dV_g, \quad v \in H^1(M).$$

Since $E(v) = \frac{1}{2}(dv, dv)_{L^2}$, the same argument as in Section 2.7 shows that any minimizer u of the Dirichlet energy functional satisfies the equation

$$\delta du = 0.$$

We have arrived at the definition of the Laplace-Beltrami operator.

Definition 3.34 (Laplace-Beltrami operator)**.** If (M, g) is a Riemannian manifold (with or without boundary), the *Laplace-Beltrami operator* is defined by

$$\Delta_g u := -\delta du.$$

The next result is immediate

Lemma 3.35. In local coordinates,

$$\Delta_g u = |g|^{-\frac{1}{2}}\partial_j\big(|g|^{\frac{1}{2}} g^{jk}\partial_k u\big),$$

where, as before, $|g| = \det(g_{jk})$ is the determinant of g.

4. A brief introduction to Hodge theory

Let (M, g) be a compact oriented Riemannian manifold with dimension $\dim M = n$. In this section we introduce a Laplace operator acting on differential forms in M, prove the Hodge decomposition for differential forms that generalizes the Helmholtz decomposition for vector fields, and study the topology of M by identifying the de Rham cohomology groups with spaces of harmonic differential forms.

Recall that we defined the Laplace-Beltrami operator Δ_g acting on scalar functions in M by looking at minimizers of the Dirichlet energy functional

$$E(u) = \int_M |du|^2 dV_g = (du, du)_{L^2}, \quad u \in H^1(M).$$

One has the trivial inequality

$$\|u\|^2_{H^1(M)} \leq E(u) + \|u\|^2_{L^2}, \quad u \in H^1(M).$$

This show that $E(u)$ controls all derivatives of u, which leads to the fact that g is an *elliptic operator*.

Now if u is a k-form in M with $k \geq 1$, we have seen two tpyes of derivatives of u: the exterior derivative $du \in \Omega^{k+1}(M)$ and also the codifferential $\delta u \in \Omega^{k-1}(M)$. We could introduce an energy functional

$$E^{(k)}(U) := (du, du)_{L^2} + (\delta u, \delta u)_{L^2}, \quad u \in H^1(M, \Lambda^k M).$$

By the *Gaffney's inequality*, this energy functional controls all first order derivatives of the k-form u.

Now, if u is a minimizer of $E^{(k)}$ in $H^1(M, \Lambda^k M)$, then for any $\varphi \in H^1_c(M, \Lambda^k M)$, we have

$$\begin{aligned}
0 &= \frac{d}{dt} E^{(k)}(u + t\varphi)|_{t=0} \\
&= \frac{d}{dt}\Big(E^{(k)}(u) + 2t\big[(du, d\varphi) + (\delta u, \delta\varphi)\big] + t^2 E^{(k)}(\varphi)\Big)|_{t=0} \\
&= \big((d\delta + \delta d)u, \varphi\big).
\end{aligned}$$

This is true for any such φ, and so a minimizer must satisfy

$$(d\delta + \delta d)u = 0.$$

Definition 4.1 (Hodge Laplacian)**.** If $0 \leq k \leq n$, we define the *Hodge Laplacian* to be the map $\Delta\colon \Omega^k(M) \to \Omega^k(M)$ satisfying

$$-\Delta = d\delta + \delta d.$$

Example 4.2. If $U \subset \mathbb{R}^3$ is an open set and $u = u^j dx^j$ is a 1-form in U, then direct computation gives

$$(d\delta + \delta d)u = (-\Delta u_j)dx^j.$$

This holds for k-forms in $\mathbb{R}^n$ as well. Namely, if $U \subset \mathbb{R}^n$ is open and if $u = u_I dx^I$ is a k-form in U, then

$$\Delta u = (\Delta u_I)dx^I,$$

where Δu_I is the Euclidean Laplacian of $u_I \in C^\infty$.

Next we study the solvability of the equation $-\Delta u = f$ on k-forms.

Definition 4.3. Let $H^{-1}(M, \Lambda^k M)$ be the dual space of $H^1(M, \Lambda^k M)$, i.e. the space of bounded linear functions on $H^1(M, \Lambda^k M)$. Given $f \in H^{-1}(M, \Lambda^k M)$, we say that $u \in H^1(M, \Lambda^k M)$ is a *weak solution* of

$$-\Delta u = f \quad \text{in } M$$

if

$$(du, dv)_{L^2} + (\delta u, \delta v)_{L^2} = f(v) \quad \text{for all } v \in H^1(M, \Lambda^k M).$$

Theorem 4.4. Fix k with $0 \leq k \leq n$.

(1) *Weak Solutions.* There is a coutable set $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}$ with

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

such that whenever $\lambda \in \mathbb{C} \backslash \{\lambda_1, \lambda_2, \cdots\}$, the equation

$$(-\Delta - \lambda) u = f$$

has a unique weak solution $u \in H^1(M, \Lambda^k M)$ for any $f \in H^{-1}(M, \Lambda^k M)$.

(2) *Kernel of $-\Delta$.* The space

$$\mathcal{H}_k := \ker \big(\Delta|_{H^1(M,\Lambda^k M)} = \{u \in H^1(M, \Lambda^k M) : \Delta u = 0\}\big)$$

is finite dimensional and its elements are C^∞.

(3) *Elliptic regularity.* There is a bounded linear map

$$G \colon L^2(M, \Lambda^k M) \to H^2(M, \Lambda^k M)$$

such that

$$-\Delta G u = (I - P_k) u, \quad u \in L^2(M, \Lambda^k M),$$

where P_k is the orthogonal projection from $L^2(M, \Lambda^k M)$ onto $\mathcal{H}_k$. For $j \geq 0$, G is a bounded map

$$H^j(M, \Lambda^k M) \to H^{j+2}(M, \Lambda^k M).$$

The finite dimensional space $\mathcal{H}_k$ is called the *space of harmonic k-forms*, and it has the following characterization.

Theorem 4.5 (Characterization of harmonic forms)**.** For $1 \leq k \leq n$, we have

$$\mathcal{H}_k = \{u \in \Omega^k(M) : du = \delta u = 0\}$$

and when $k = 0$, it holds

$$\mathcal{H}_0 = \{u \in C^\infty(M) : u \text{ is constant on each component of } M\}.$$

In particular, $\dim(\mathcal{H}_0)$ is the number of connected component of M.

Proof. content... □

The next result is a powerful generazation of the Helmhotz decomposition, which allows to decompose a vector field F in $\mathbb{R}^n$ into curl-free and divergence-free components, i.e.

$$F = \nabla p + W,$$

where p is a scalar function and $\nabla \cdot W = 0$. The Helmholtz decomposition corresponds to the next theorem in the case of 1-forms.

Theorem 4.6 (Hodge decomposition)**.** Any $u \in L^2(M, \Lambda^k M)$ admits a decomposition

$$u = d\delta G u + \delta d G u + P_k u,$$

where the three components are L^2-orthogonal.

The Hodge decomposition of $u \in L^2(M, \Lambda^k M)$ can also be written as

$$u = d\alpha + \delta\beta + \gamma,$$

where $\alpha = \delta G u \in H^1(M, \Lambda^{k-1} M)$, $\beta = dGu \in H^1(M, \Lambda^{k+1} M)$ and $\gamma = P_k u \in \mathcal{H}_k$ is a harmonic k-form (and hence C^∞).

Proof of Theorem 4.6. Let $u \in L^2(M, \Lambda^k M)$. By Theorem 4.4, we have

$$-\Delta(Gu) = (I - P_k)u.$$

The decomposition follows since $-\Delta = d\delta + \delta d$. The orthogonality follows since

$$(d\alpha, \delta\beta)_{L^2} = (d^2\alpha, \beta)_{L^2} = 0$$

and since any harmonic form γ is L^2-orthogonal to any $d\alpha$ or $\delta\beta$ because of the equation

$$d\gamma = \delta\gamma = 0.$$

□

Let M be a compact smooth manifold. We define the *de Rham cohomology groups* for $0 \leq k \leq n$ by

$$H^k_{\mathrm{dR}}(M) := \ker(d|_{\Omega^k(M)}) / \operatorname{Im}(d|_{\Omega^{k-1}(M)}).$$

These are vector spaces. If $F\colon M \to N$ is a diffeomorphism between two compact smooth manifolds, the property $dF^* = F^* d$ immediately implies that F^* induces an isomorphism between the vector space $H^k_{\mathrm{dR}}(N)$ and $H^k_{\mathrm{dR}}(M)$. Thus the de Rham cohomology groups are *diffeomorphic invariants*, it is not hard to show that they are actually *topological* and even *homotopy* invariants and hence do not depend on the particular smooth structure that M has.

The next theorem, due to Hodge, shows that if one assigns a Riemannian metric g on M, then $H^k_{\mathrm{dR}}(M)$ can be identified with the space of harmonic k-forms. This shows, in particular, that the dimension of $\mathcal{H}_k$ is independent of g and in fact is a topological invariant.

Theorem 4.7 (Hodge isomorphism)**.** If $0 \leq k \leq n$, then any equivalence class in $H^k_{\mathrm{dR}}(M)$ has a unique harmonic representative. The map

$$\mathcal{J}_k \colon \mathcal{H}_k \to H^k_{\mathrm{dR}}(M), \quad u \mapsto [u]$$

is an isomorphism.

Proof. Let $\omega \in \Omega^k(M)$ satisfy $d\omega = 0$, and let $[\omega] \in H^k_{\mathrm{dR}}(M)$ be the corresponding equivalence class. We need to show that $[\omega] = [u]$ for a unique $u \in \mathcal{H}_k$. To show the existence, write the Hodge decomposition for ω:

$$\omega = d\delta G\omega + \delta dG\omega + P_k\omega.$$

Note that $d\omega = 0$, it follows that $(\omega, d\alpha) = 0$ for all α, and in particular,

$$0 = (\omega, \delta dG\omega) = (d\delta G\omega + \delta dG\omega + P_k\omega, \delta dG\omega) = \|\delta dG\omega\|^2.$$

Thus $\delta dG\omega = 0$, which implies that

$$\omega = u + d\delta G\omega,$$

where $u = P_k\omega$ is harmonic. This shows that $[\omega] = [u]$ for some harmonic u. To show the uniqueness, we note that if $[u_1] = [u_2]$ with u_i harmonic for $i = 1, 2$, then $u_1 - u_2 = d\alpha$ for some α, but then

$$\|u_1 - u_2\|^2 = (u_1 - u_2, d\alpha) = \big(\delta(u_1 - u_2), \alpha\big) = 0,$$

which implies that $u_1 = u_2$. The fact that $\mathcal{J}_k$ is an isomorphism follows immediately from the above facts. $\square$

As an immediate consequence, we have the following corollary.

Corollary 4.8 (Betti numbers)**.** Let M be a compact oriented smooth manifold. The de Rham cohomology groups of M are finite dimensional vector spaces, and their dimensions are given by

$$b_k(M) := \dim\big(H^k_{\mathrm{dR}}(M)\big) = \dim\big(\ker(\Delta_g|_{\Omega^k(M)})\big),$$

where g is any Riemannian metric on M.

Next we discuss the Poincaré duality, which states that there is a natural isomorphism between $H^k_{\mathrm{dR}}(M)$ and $H^{n-k}_{\mathrm{dR}}(M)$ whenever $0 \le k \le n$. In terms of Betti numbers, this implies that $b_k(M) = b_{n-k}(M)$. The isomorphism is given by the Hodge star operator.

Theorem 4.9 (Hodge star operator)**.** Let (M, g) be an oriented Riemannian manifold of dimension n. There is a unique linear operator, called the *Hodge star operator*,

$$*\colon \Omega^k(M) \to \Omega^{n-k}(M)$$

which satisfies the following identity for $u, v \in \Omega^k(M)$

$$u \wedge *v = \langle u, v\rangle dV. \tag{4.1}$$

It has the following properties:

(1) $** = (-1)^{k(n-k)}$ on k-forms
(2) $*1 = dV$
(3) $*(\varepsilon^1\wedge\cdots\wedge\varepsilon^k) = \varepsilon^{k+1}\wedge\cdots\wedge\varepsilon^n$, whenever $(\varepsilon^1, \cdots, \varepsilon^n)$ is a positive local orthonormal frame on T^*M
(4) The codifferential has the expression

$$\delta = (-1)^{(k-1)(n-k)-1} * d * \quad \text{on } k\text{-forms.}$$

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