Correction Serie 11/12

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Exercise 7. Soit s la symétrie par rapport au plan affine d'equation

$$P:\ x+2y+3z=4.$$

1. Calculer la matrice (dans la base canonique) de la partie linéaire s_0 de s et le vecteur $\vec{v}$ tel que

$$s=t_{\vec{v}}\circ s_0.$$

Denote by P_0 the plane that passes through the origin,

$$P_0:\ x+2y+3z=0.$$

Let v_1 and $v_2\in\mathbb{R}^3$ be two vectors that lie in the plane, such that v_1 is orthogonal to v_2. Choose for example

$$v_1=\begin{pmatrix}3\\0\\-1\end{pmatrix},\quad v_2=\begin{pmatrix}1\\-5\\3\end{pmatrix}.$$

Normalizing yields

$$v_1'=\frac{1}{\sqrt{10}}\begin{pmatrix}3\\0\\-1\end{pmatrix},\quad v_2'=\frac{1}{\sqrt{35}}\begin{pmatrix}1\\-5\\3\end{pmatrix}.$$

Let v_3 be a third vector in $\mathbb{R}^3$ that lies orthogonal to the other two vectors. Choose

$$v_3=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

with normalization

$$v_3'=\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

We obtain an orthonormal basis $\mathcal{B} = \{v_1', v_2', v_3'\}$ for which the matrix of the symmetry s_0 is equal to

$$M_{s_0,\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In order to express the matrix of s_0 in the canonical basis $\mathcal{B}_0 = \{e_1, e_2, e_3\}$, we let A be the base change matrix from $\mathcal{B}$ to $\mathcal{B}_0$, that contains the coefficients of the vectors v_i' in the basis $\mathcal{B}_0$ in its columns,

$$A = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{pmatrix}.$$

It holds that

$$M_{s_0,\mathcal{B}_0} = A * M_{s_0,\mathcal{B}} * A^t = \frac{1}{7} \begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix}.$$

It remains to describe the translation vector t_v for which $s = t_v \circ s_0$ holds. This is a vector that is orthogonal to the plane P_0, respectively P. Hence v is of the form $(\alpha, 2\alpha, 3\alpha)$. The point of this form that is contained in the plane P is equal to

$$\alpha + 2(2\alpha) + 3(3\alpha) = 4 \Rightarrow \alpha = \frac{2}{7}.$$

This point $Q = (\alpha, 2\alpha, 3\alpha) = (\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$ needs to stay fixed under the action of s. Hence if we write $s = t_v \circ s_0$, we note that Q gets sent to $-Q$ by s_0, and hence we need to apply two tranlations $t_{(\frac{2}{7},\frac{4}{7},\frac{6}{7})}$ for $-Q$ to be sent back to Q. Summing up, we get $t_v = t_{(\frac{4}{7},\frac{8}{7},\frac{12}{7})}$.

2. Que vaut s^{2019}?

   Since s is simply a symmetry along a plane, applying s twice gives back the identity, and hence applying s 2019 times gives back s.

3. Même question pour la composée $s' = t_{(1,1,1)} \circ s$.

   It holds that

$$s' = t_{(1,1,1)} \circ s = t_{(1,1,1)} \circ t_v \circ s_0,$$

   where the vector v is a vector orthogonal to the plane of symmetry by the first question. We want to decompose the translation $t_{(1,1,1)}$ into a translation by a vector u that is orthogonal to the plane of symmetry and a translation by a vector w that lies in the plane of symmetry. Then

$$s' = t_w \circ \underbrace{t_u \circ t_v \circ s_0}_{\tilde{s}} = t_w \circ \tilde{s},$$

where $\tilde{s}$ is a symmetry along a plane and $\tilde{s}$ commutes with the translation t_w.

In order to decompose the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ into the two vectors u and w, we write $\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha v_1 + \beta v_2 + \gamma v_3$, where v_1 and v_2 form a basis of the plane P_0, and v_3 is a vector orthogonal to it. We get $\alpha = \frac{1}{5}, \beta = -\frac{1}{35}$ and $\gamma = \frac{3}{7}$, and hence

$$w = \alpha v_1 + \beta v_2 = \begin{pmatrix}\frac{4}{7}\\\frac{1}{7}\\-\frac{2}{7}\end{pmatrix} \quad u = \gamma v_3 = \begin{pmatrix}\frac{3}{7}\\\frac{6}{7}\\\frac{9}{7}\end{pmatrix}.$$

So

$$s' = t_{(\frac{4}{7},\frac{1}{7},-\frac{2}{7})} \circ \underbrace{t_{(\frac{3}{7},\frac{6}{7},\frac{9}{7})} \circ t_{(\frac{4}{7},\frac{8}{7},\frac{12}{7})} \circ s_0}_{=:\tilde{s}}.$$

Since t_w and $\tilde{s}$ commute, it holds that

$$(s')^{2019} = (t_w \circ \tilde{s})^{2019} = t_w^{2019} \circ (\tilde{s})^{2019} = t_{2019*(\frac{4}{7},\frac{1}{7},-\frac{2}{7})} \circ \tilde{s}.$$

Exercise 8. Soient $a, b, c, d, e, f \in \mathbb{R}$. On considère la transformation de l'espace donnée dans la base canonique par $\varphi(x, y, z) = (X, Y, Z)$ avec

$$\begin{aligned} X &= \frac{1}{d}(2x - 2y + az) + 1 \\ Y &= \frac{1}{d}(x + by + 2z) + e \\ Z &= \frac{1}{d}(cx - y + 2z) + f \end{aligned}$$

1. Décrire l'ensemble des $(a, b, c, d, e, f) \in \mathbb{R}^5$ tels que φ est un vissage.

   Let $\varphi = t_v \circ \varphi_0$, where φ_0 is the linear part, which is a rotation in the case of a vissage, and t_v is a translation by a vector that is not orthogonal to the axis of rotation. The matrix corresponding to the linear part is

   $$M_{\varphi_0} = \frac{1}{d}\begin{pmatrix}2 & -2 & a\\1 & b & 2\\c & -1 & 2\end{pmatrix},$$

   and the translation vector is

   $$t_v = \begin{pmatrix}1\\e\\f\end{pmatrix}.$$

In order for φ_0 to be a rotation, this matrix needs to be orthogonal and of determinant 1. The ortogonality condition poses the following restrictions on the variables a, b, c, d, e, f :

$$\frac{1}{d^2}\begin{pmatrix} 2 & -2 & a \\ 1 & b & 2 \\ c & -1 & 2 \end{pmatrix} * \begin{pmatrix} 2 & 1 & c \\ -2 & b & -1 \\ a & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that

- $8 + a^2 = d^2$
- $2 - 2b + 2a = 0$
- $2c + 2 + 2a = 0$
- $2 - 2b + 2a = 0$
- $5 + b^2 = d^2$
- $c - b + 4 = 0$
- $2c + 2 + 2a = 0$
- $c - b + 4 = 0$
- $c^2 + 5 = d^2$

From these equations we get

$$a = 1, b = 2, c = -2, d = \pm 3.$$

Using the fact that the determinant of M_{φ_0} needs to be 1, it follows that $d = +3$. The axis of rotation is defined to be

$$D_0 = \ker(\varphi_0 - \mathrm{Id}) = \ker\left(\frac{1}{3}\begin{pmatrix} -1 & -2 & 1 \\ 1 & -1 & 2 \\ -2 & -1 & -1 \end{pmatrix}\right)$$

The vector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is contained in the kernel, and hence $D_0 = \mathbb{R}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. The isometry φ is a vissage if the vector v, by which we translate is not orthogonal to D_0, i.e. if $-1 + e + f \neq 0 \Rightarrow e + f \neq 1$.

2. Même question en demandant que φ soit une anti-rotation.

   The only difference to the first part is that in the case of an anti-rotation, the determinant of M_{φ_0} is equal to -1 and so $d = -3$. The translation vector can be any vector, and the axis of rotations stays the same.

3. Si φ n'est pas un vissage montrer que $\varphi^6 = \mathrm{Id}_{\mathbb{R}^3}$.

   If φ is no vissage, it is either a rotation or an anti-rotation. Consider the case of a rotation first. The angle θ can be determined via

$$\cos\theta = \frac{1}{2}(\mathrm{tr}(M_{\varphi_0}) - \det(M_{\varphi_0})) = \frac{1}{2}(2-1) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

   Hence rotating 6 times results in the identity.

   In case of an antirotation, we get

$$\cos\theta = \frac{1}{2}(\mathrm{tr}(M_{\varphi_0}) - \det(M_{\varphi_0})) = \frac{1}{2}(-2-(-1)) = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}.$$

   Hence applying this 6 times results in the identity.