List of problems for final oral exams

In the final exam, it might happen a few problems listed before were regarded as one problem. For instance, we may have a Problem 1 in the final oral exam, which contains Problems 1 and 2 below together (as just one problem).

The total time of the oral exam will be half an hour without preparation. We have several different exam sheets, and each exam sheet contains 6 combined problems from the following list of problems. Each of the 6 problems on each sheet counts 1 point.

1. Chapter 2

1. Write down the Taylor expansion (as in Theorem 2.1) and state the fundamental theorem of calculus for C^1-functions $f\colon [a,b] \to \mathbb{R}$. Sketch the proof of it.

2. Write down the Taylor expansion, multi-index version (as in Theorem 2.3). Sketch the proof as in the exercise.

3. Give the definition of an m-tensor field in an open set $U \subset \mathbb{R}^n$ and a symmetric m-tensor field as in Definition 2.4

4. Give examples of 0, 1, 2 tensors and the way to generate an m-tensor from a smooth function f (Example 2.6)

5. Give the definition of a multlinear map (as in Definition 2.7)

6. State the theorem for tensors as multilinear maps (as in Theorem 2.8), alternatively, you can explain this theorem for the case of 2-tensors as in Page 6 bottom part.

7. State the definition of a vector field in an open set $U \subset \mathbb{R}^n$ and the gradient vector field of a smooth function $u \in C^\infty(U)$

8. Prove Lemma 2.11 (we will remind you the definition of curl and divergence)

9. State the equations (2.2) and (2.3) clearly

10. State the definition of differential forms as in Definition 2.13 and remember Remark 2.14

11. State the definition of an exterior derivative (as in Definition 2.15)

12. Remember the examples in Example 2.16

13. State and prove Lemma 2.17 about the exterior derivatives in two and three dimensions

14. State the definition of an alternating tensor field in an open set $U \subset \mathbb{R}^n$ (as in Definition 2.19) and prove Lemma 2.18 (we will state the lemma)

15. State the definition of exact forms and closed forms and explain their relations with examples

16. State the definition of the de Rham cohomology groups of an open set $U \subset \mathbb{R}^n$ (as in Definition 2.21)

17. State the Poincaré lemma as in Lemma 2.22 and sketch the proof in dimension 2.

18. Given the definition of tangent space and tangent bundle as in Definition 2.23

19. State the definition of a Riemannian metric (as in Definition 2.24)

20. State the definition of the length of a tangent vector and angle between two tangent vectors.

21. State the definition of a regular curve and its length for (U, g) (as in Definition 2.25); prove that the length of a regular curve is independent of the parametrization.

22. State the definition of Riemannian distance between two points in a connected open set U

23. State the definition of a length minimizing curves and the definition of a geodesic (you do not need to write down the geodesic equation, but mention it as a second oder ODE involving Christoffel symbols of g)

24. State the relationship between length minimizing curves and geodesics, i.e. are length minimizing curves geodesic? are geodesics length minimizing curves? Give examples to support the statement.

25. State the Hopf-Rinow theorem

26. Describe the idea of proof of Theorem 2.26, i.e., how to deduce equations from variationally minimizers? You just need to describe the first paragraph under the section **Variations of curves** on page 16 of the notes.

27. State the definition of Riemannian volume and integration (as in Definition 2.33) and explain the motivation why it is defined like this (not necessarily the detailed proofs)

28. State the definition of L^2-inner product for functions, vector fields and 1-forms (as in Definition 2.35, Equation (2.9) and (2.10))

29. State Theorem 2.38 for codifferentials (you do not need to write (2.12))

30. State the definition of the Laplace-Beltrami operator (as in Definition 2.39)

31. State the definition of an exterior derivative (as in Definition 2.7) and explain its relation with the Dirichlet energy (i.e. you need to state what is the Dirichlet energy of a smooth map and then show that minimizers of the Dirichlet energy satisfies the harmonic equation $\Delta_g u = 0$)

2. Chapter 3

32. State the definition of a smooth manifold (as in Definition 3.1) and also remember the terminology such as atlas, charts and local coordinate. State the definition of a smooth map (as in Definition 3.3) between two manifolds.

33. State the definition of a derivation (as in Definition 3.4) and give an example of derivation.

34. Give a basis of the tangent space T_pM and state the definition of the tangent bundle TM. More over, give the standard charts of TM and state the definition of vector fields on M (as in Definition 3.5)

35. State the definition of the tensor bundle T^kM of M and give the definition of differential k-forms as in Definition 3.9

36. Express vector fields, k-tensor fields, differential 1-form and differential k-forms in local coordinate. Give examples of vector fields, 1-forms, 2-tensor fields and n-forms on an n-dimensional Riemannian manifolds (give one example for each class)

37. State the definition of pushforward of a vector field (as in definition 3.12) and State the definition of pullback of k-tensor fields (as in Definition 3.15).

38. Prove Lemma 3.13 and Lemma 3.17 (we shall state the lemmas)

39. State the definition of partition of unity (as in Lemma 3.20) and explain how to use this to give the definition of integration of a compactly supported n-form on an n-dimensional manifold

40. State the definition of a Riemannian metric and give the expression of a Riemannian metric in local coordinates. Moreover, exlain hwo to construct a Riemannian metric on a smooth manifold (only indicate the idea of the proof of Theorem 3.26, you can draw pictures to indicate the idea)

41. State the definition of an immersed manifold (as in Example 3.25 (5)) and verify it.

42. How to define the Riemannian distance on a Riemannian manifold and explain what Theorem 3.27 means (not give a proof, but just explain the meaning of the statement in Theorem 3.27).

43. State the definition of Riemannian isometry between two Riemannian manifolds and the definition of distance-preserving homeomorphism between two metric spaces. State the relation of these two isometries in the setting of Riemannian manifolds (i.e. State Proposition 3.29 and Theorem 3.30). Prove Proposition 3.29.

44. Explain on a Riemannian manifold, there is a canonical way of converting tangent vectors into cotangent vectors and vice versa.

45. State the definition of an orthonormal frames and explain briefly how to obtain a local orthonormal frame in some neighborhood of each point $p \in M$. Express the Riemannian metric g in local frame.

46. State the definition of a volume form on a Riemannian manifold and give the expression in local coordinate. Explain why the volume form is nature (in terms of orientation preserving diffeomorphisms).

47. Use the Hodge decomposition to prove the Hodge isomorphism theorem and state the Betti number corollary.

48*[1]. State the definition of Hodge star operator (as in Theorem 4.9) and the Poincaré duality.

[1] If we are not able to cover this part, then this will not appear in the final exam