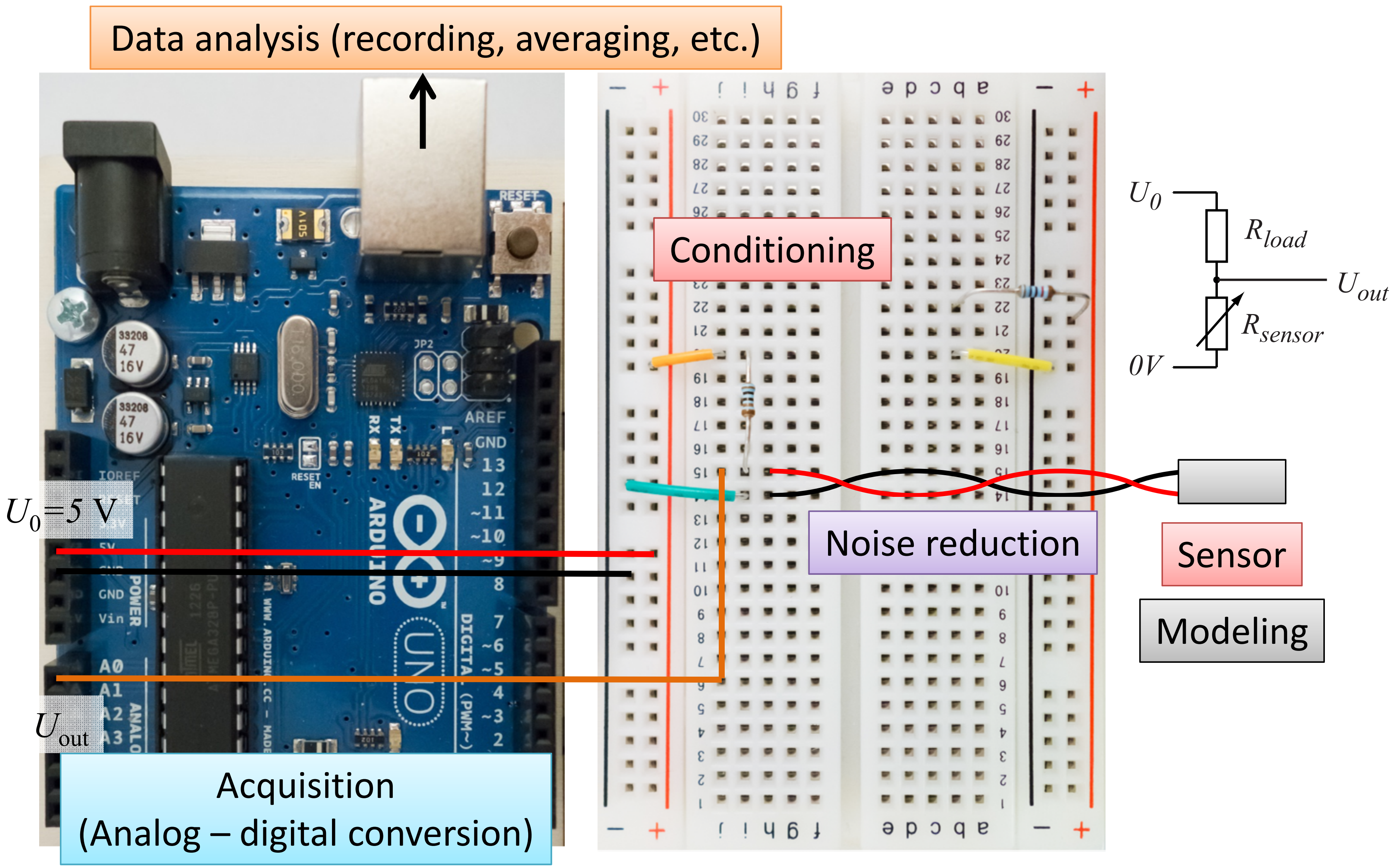
Measurement systems

Lecturer: Andras Kis

Chapter 5: Data Analysis

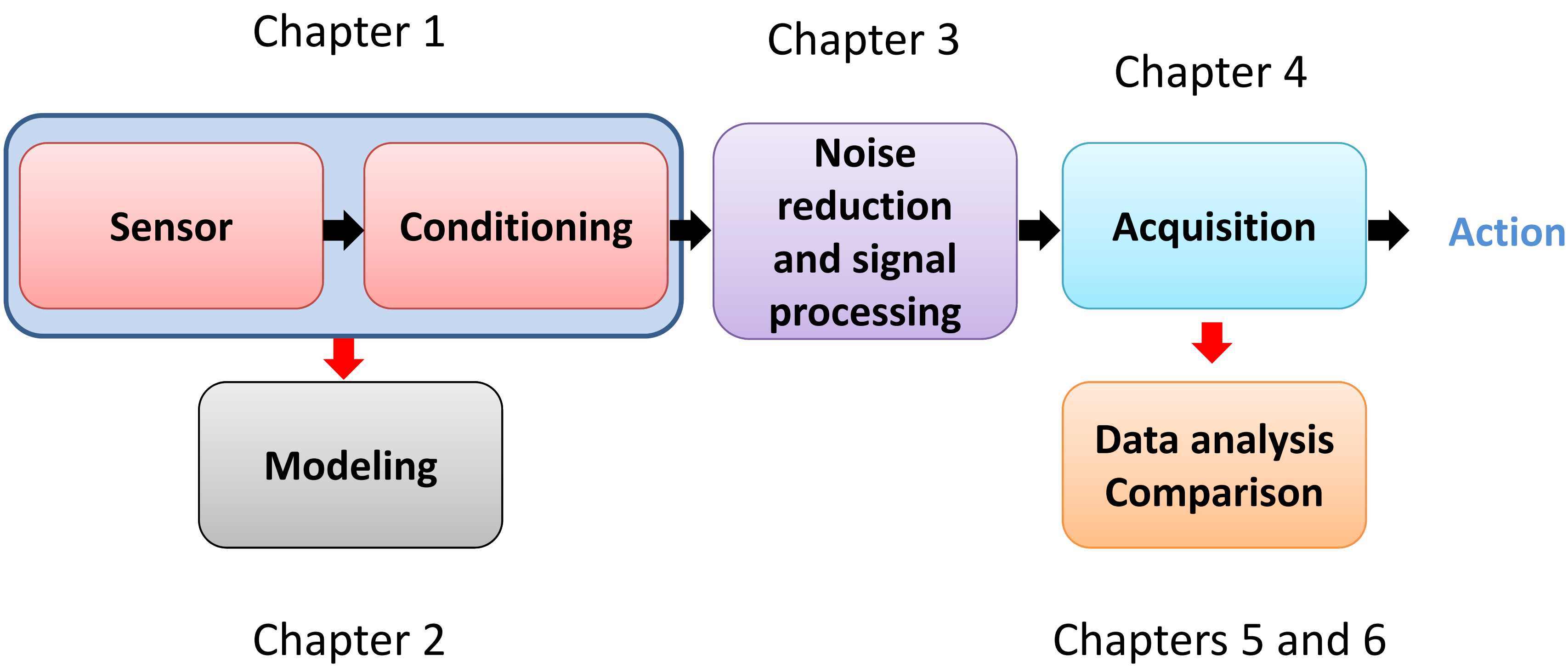
Measurement chain



Arduino UNO board

Conditioning circuit

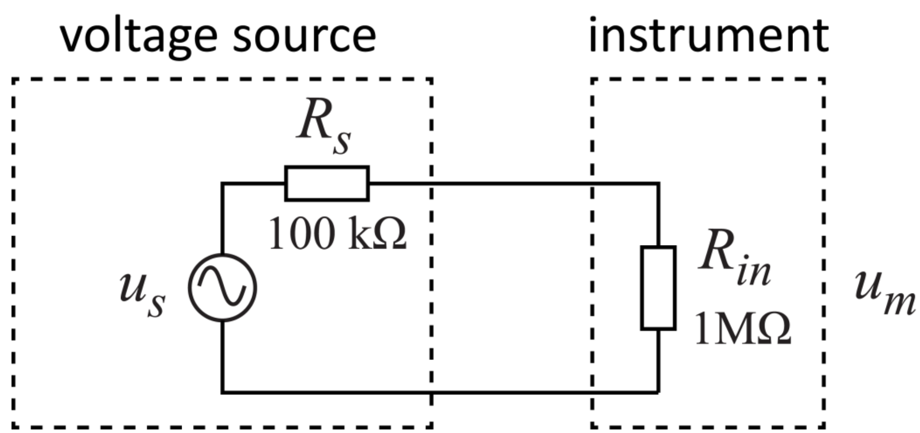
Measurement chain



The attributes of error

- Let x_0 be the real value and x the measured value
- Error:
  - Absolute $\Delta x = x - x_o$
  - Relative $\Delta_r x = \dfrac{x - x_o}{x_0}$
- Nature of the error
  - Systematic – known origin, is repeatable, can be corrected
  - Random – stochastic phenomena (noise), cannot be corrected
- Magnitude of the error
  - Maximal error – absolute limits
  - Probable error – limits associated with a given probability

Systematic error - example



- Voltage measurement:
  - R_s – internal resistance of the voltage source (sensor)
  - R_{input} – input resistance of the instrument (voltmeter)

$$u_m = \frac{R_{in}}{R_{in} + R_s} u_s =$$

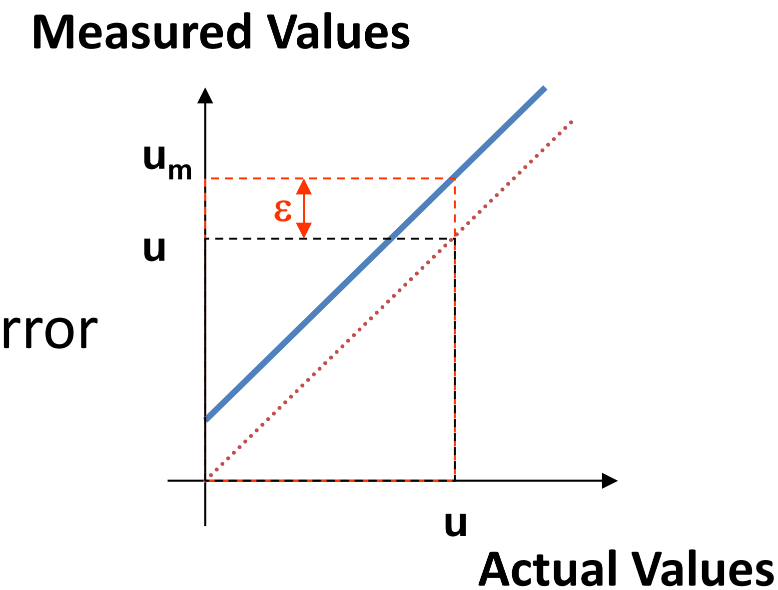
$$= \frac{1\mathrm{M\Omega}}{100\mathrm{k\Omega} + 1\mathrm{M\Omega}} = 0.91 \times u_s$$

Systematic error - example

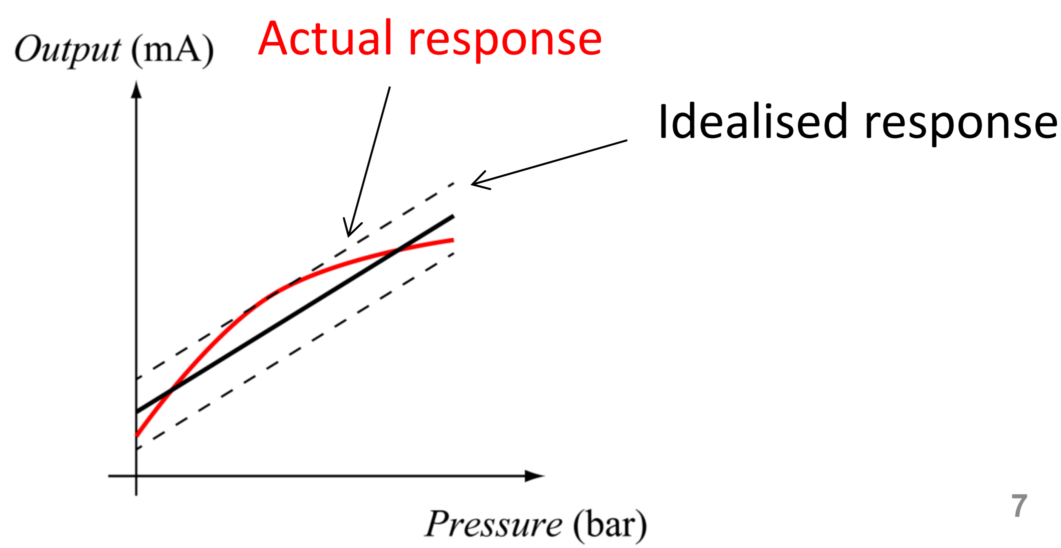
- Offset

$$u_m = u + \varepsilon$$

ε: systematic error



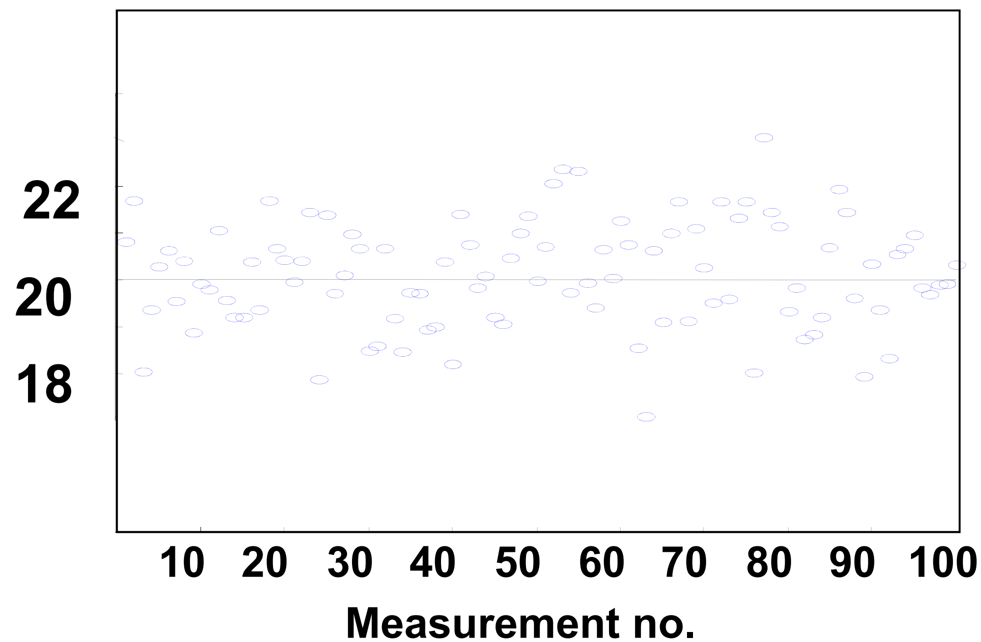
- Non-linearity
  - Known and can be corrected



Examples of random error

- Electronic noise
- Interference: EM fields, ground loops
- Temperature changes, motion of conductors
- Drift
- Hysteresis
- Repeatability

Example: temperature in the room



Error estimate

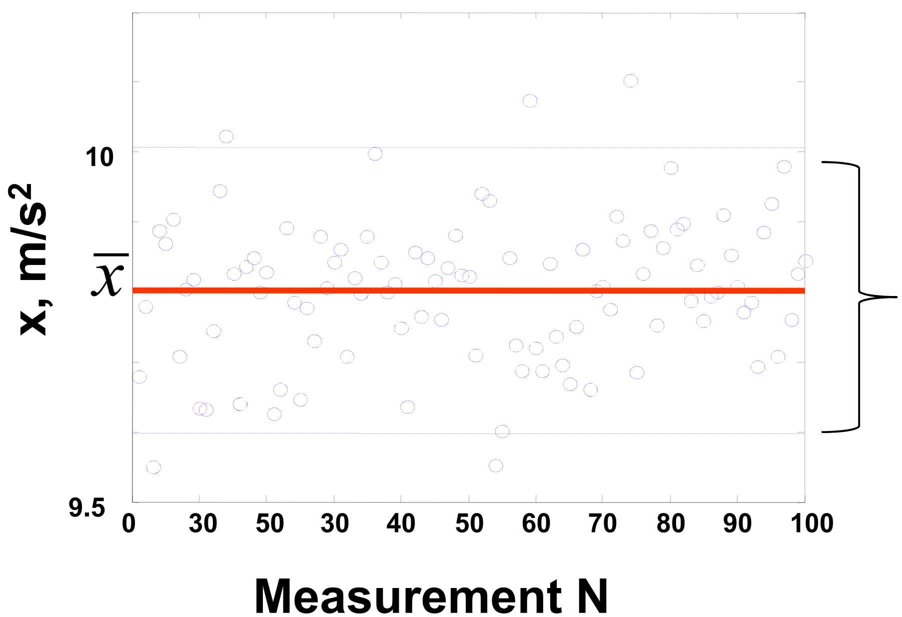
- Reporting a measurement result

$$x = \bar{x} \pm \Delta x \; (p = p_o)$$

$\bar{x}$ Central tendency (usually average, mean value)

Δx Incertitude

p probability that the value x is in the range $\bar{x} - \Delta x, \bar{x} + \Delta x$



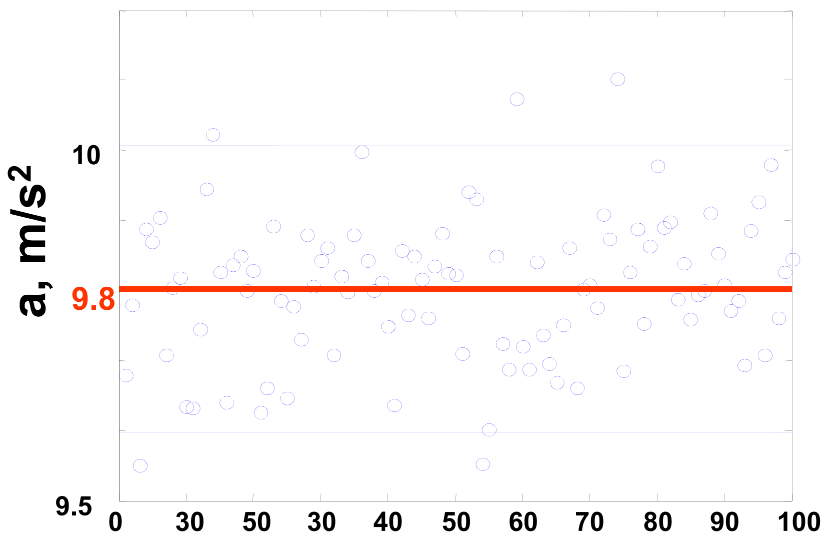
How to estimate the error:
- Collect samples
- Estimate the central tendency
- Estimate the dispersion interval
- Estimate the incertitude and its probability

Estimating the central tendency

- Average (arithmetic mean)

$$\overline{x} = \frac{1}{N}\sum_i x_i$$

$$\overline{a} = \frac{1}{N}\sum_i a_i = 9.8m/s^2$$



Other estimates of the central tendency

- Geometric mean
  - Sensitive to extreme values

$$m_g = \left(\prod_{i=1}^{N} x_i \right)^{\frac{1}{N}}$$

- Harmonic mean
  - Sensitive to extreme values

$$m_h = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$$

- Median
  - Less sensitive to extreme values

  Sort x_i (measurement results) from lowest to highest

$$Median = \frac{x_{N/2} + x_{N/2+1}}{2} \quad N \text{ even}$$

$$Median = x_{N/2} \quad N \text{ odd}$$

Example:

Average (2,2,3,4,3,10,10)=4.9

Median (2,2,3,4,3,10,10)= Median (2,2,3,3,4,10,10) = 3

Example

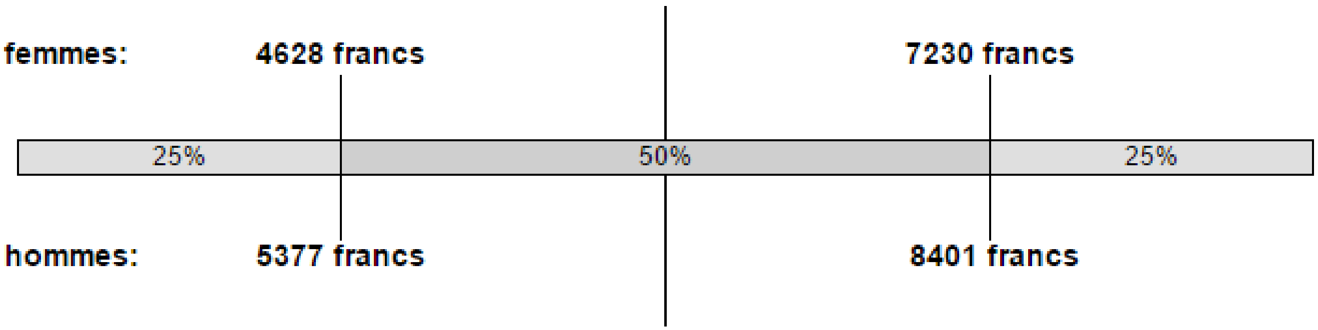
- http://www.lohnrechner.bfs.admin.ch/Pages/SalariumWizard.aspx?lang=fr

Branche économique :	72..Recherche-développement scientifique *
Région :	Région lémanique (VD, VS, GE) * ?
Activité :	28..Recherche et développement * ?
Niveau de qualification :	Travaux les plus exigeants et les plus difficiles * ?
Position professionnelle :	Cadre inférieur * ?
Temps de travail (heures) :	
Taux d'occupation [1 - 100%]	100 ?
Horaire hebdomadaire pour un poste à plein temps [par ex. 41.50]	41.5
	OU
Votre horaire hebdomadaire [par ex. 32.80]	41.50 *
Formation :	Haute école universitaire (UNI, EPF) ?
Age :	26
Années de service :	0 ?
Statut de séjour :	Suisse ?
Taille de l'entreprise :	50 employés et plus
L'entreprise paie-t-elle 12 ou 13 salaires mensuels ?	12 salaires mensuels ?
Touchez-vous des paiements spéciaux ?	Non ?
Etes-vous payé/e à l'heure ou au mois ?	Salaire mensuel

Salaire mensuel brut (médiane) des femmes: **5772 francs**

(41.50 h / semaine)

Dispersion des salaires chez les femmes: **4628 francs** **7230 francs**



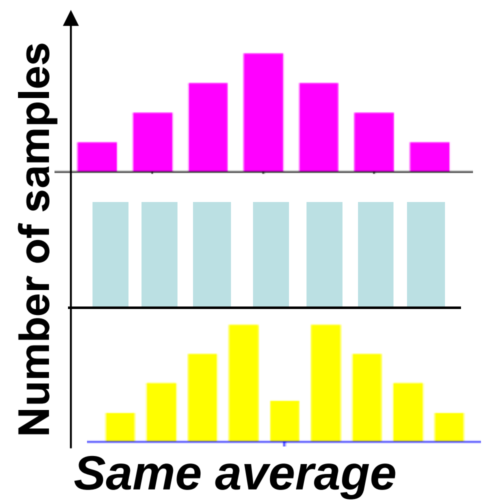
Dispersion des salaires chez les hommes: **5377 francs** **8401 francs**

Salaire mensuel brut (médiane) des hommes: **6706 francs**

(41.50 h / semaine)

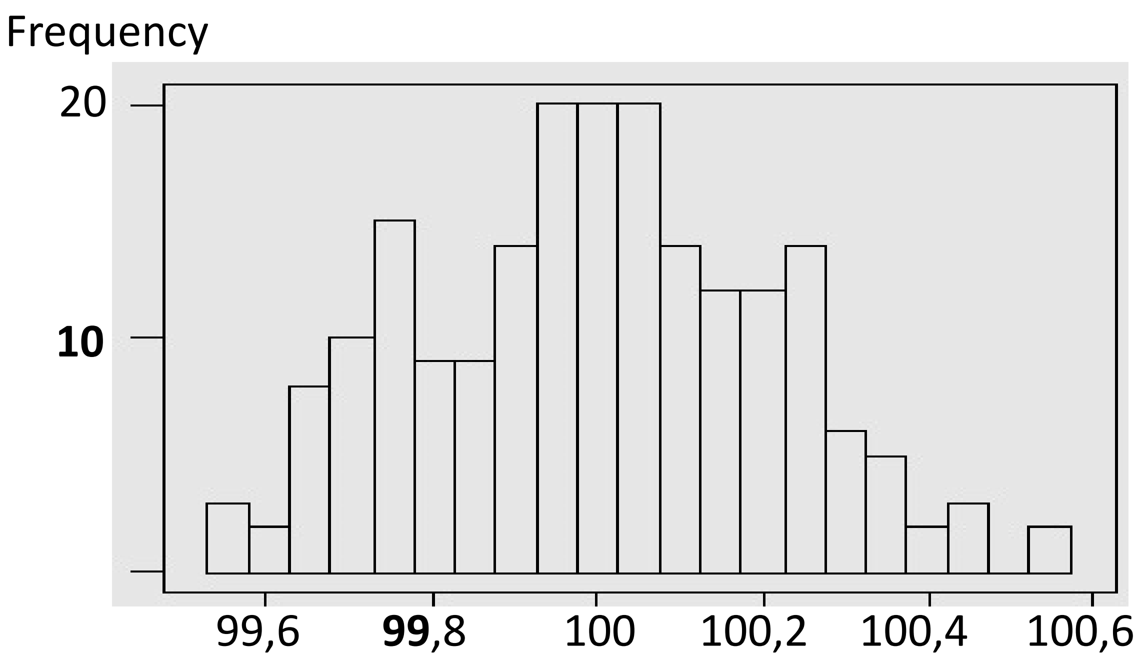
Distribution of measurements

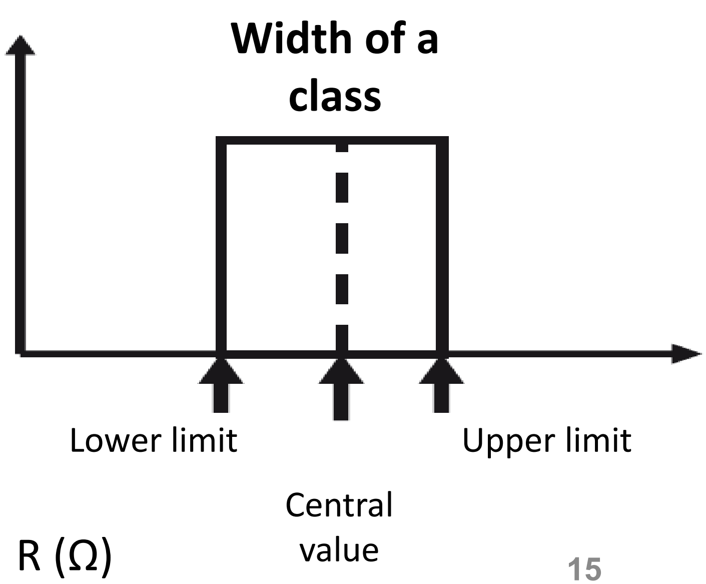
- Central tendency
  - No information on the distribution
- Histogram
  - Graphical representation of the distribution of measurement results:
    - Central, peak value
    - Range of values
    - Dispersion
    - Presence of extreme values
    - Distribution profile



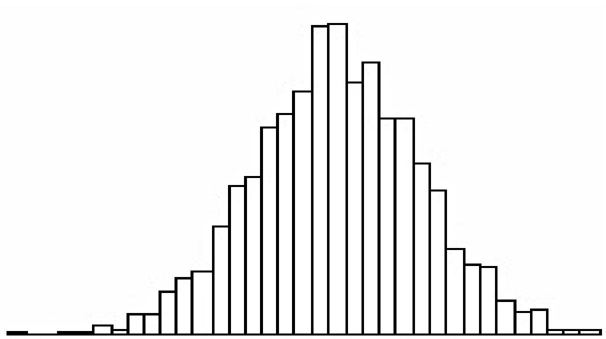
Histogram - example

- 400 samples with a resistance of 100 Ω are measured using an ohm-meter
  - The range of results is divided in k classes (bins): **x axis**
  - Number of results in each class (bin): **y axis**
  - Number of samples: N
  - Good choice for the number of classes: $k = \sqrt{N}$

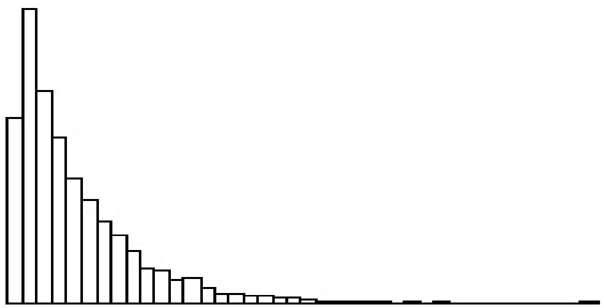




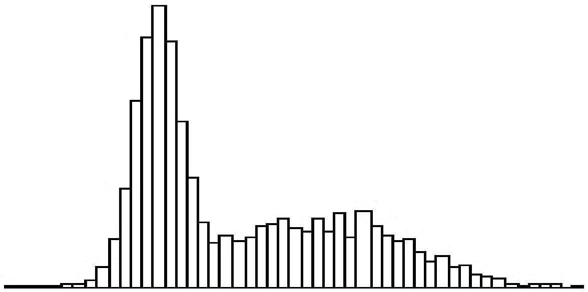
Examples of histograms



Symmetric: walking speed of young people



Nonsymmetric: duration of walking periods during the day



Symmetric: walking speed of young people and seniors

Estimate of dispersion

- Variance $$Var(x) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2$$

- Standard deviation $$\sigma = \sqrt{Var(x)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

- For N<30 or a limited number of measurements

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

- Other estimates of dispersion:

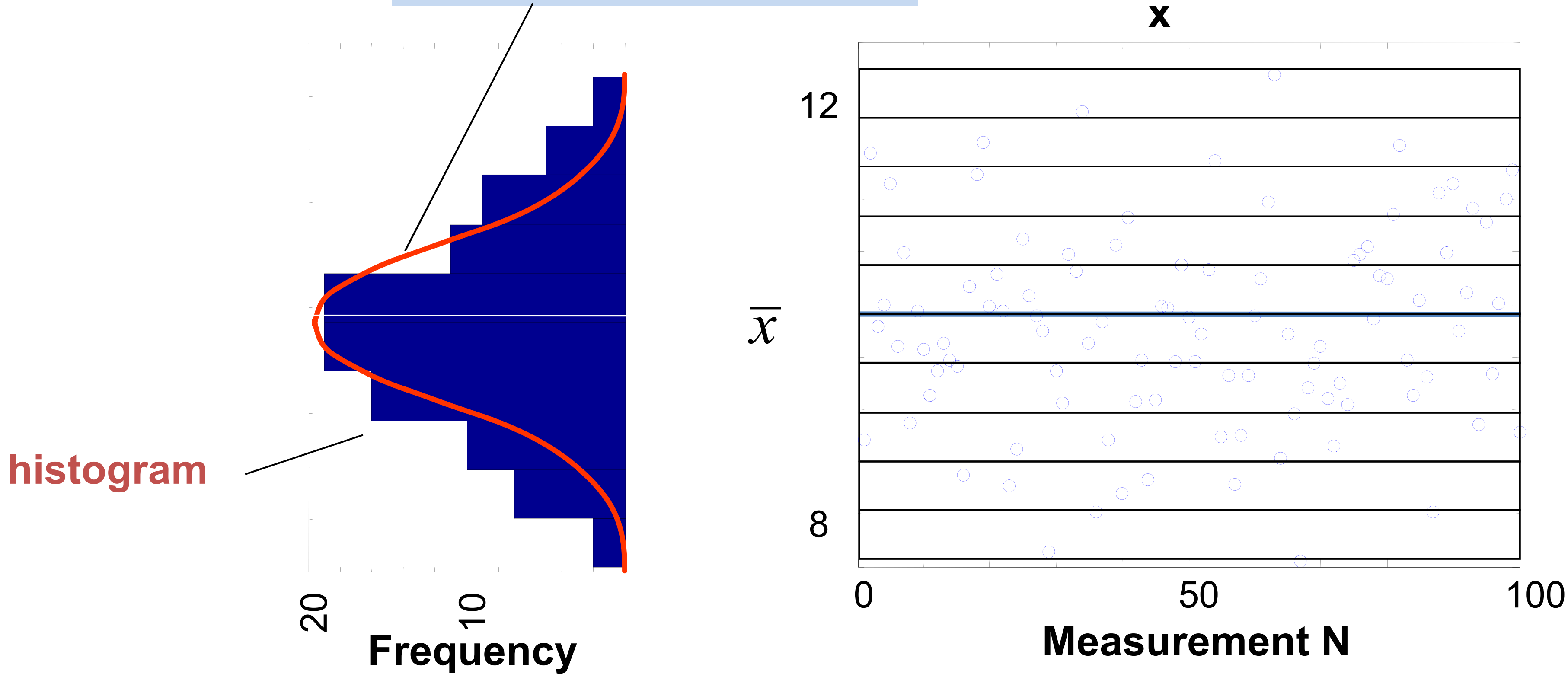
  - z score $$z = \frac{x - \overline{x}}{\sigma}$$

  - Range: x_{max} - x_{min}

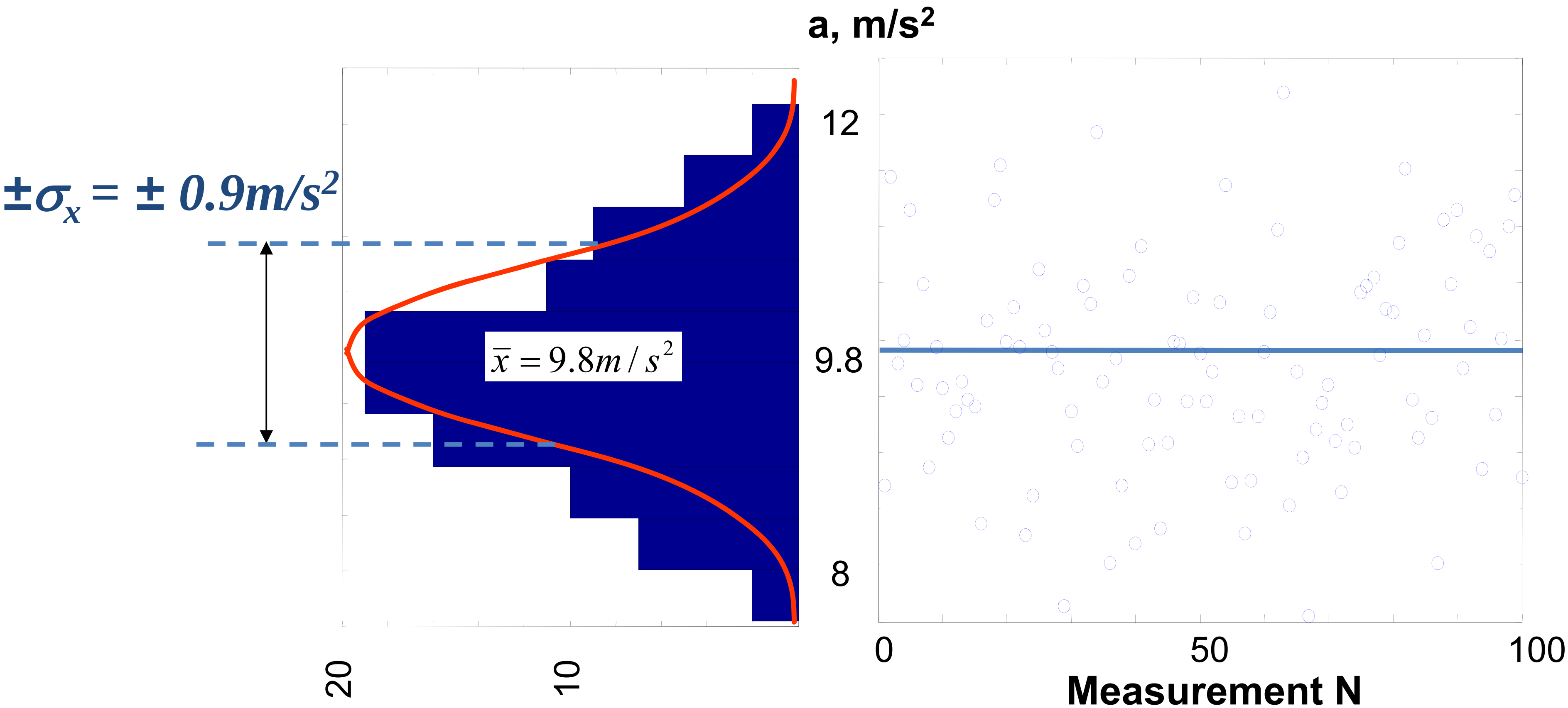
Model for dispersion

- Normal distribution (Gaussian distribution)

frequency $$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2}$$



Measuring dispersion



$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 = 0.81\, m^2/s^4$$

$$\sigma_x = 0.9\, m/s^2$$

$$f(x) = \frac{1}{0.9\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-9.8}{0.9}\right)^2}$$

Source: wikipedia

Error Probability

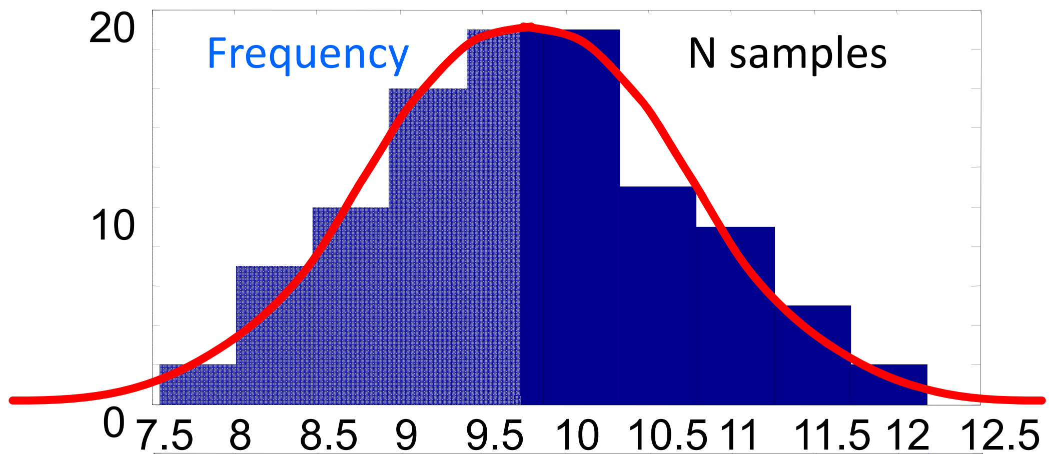
- Frequency

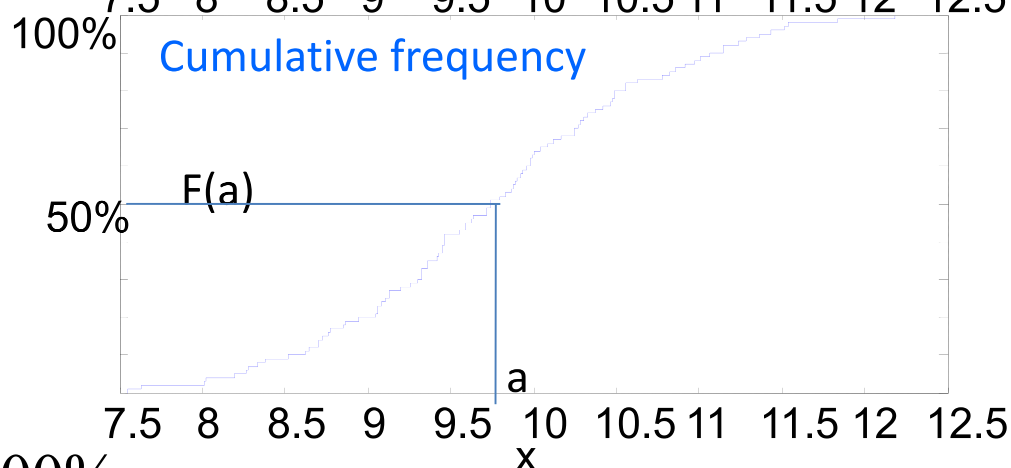
$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2}$$

- Cumulative frequency

$$F(x) = \int_{-\infty}^{x} f(x')dx'$$

- Probability





$$P(-\infty < x < \infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 = 100\%$$

$$P(x \le a) = \int_{-\infty}^{a} f(x)dx = F(a)$$

$$P(b \le x \le a) = \int_{b}^{a} f(x)dx = F(a) - F(b)$$

Standard normal distribution

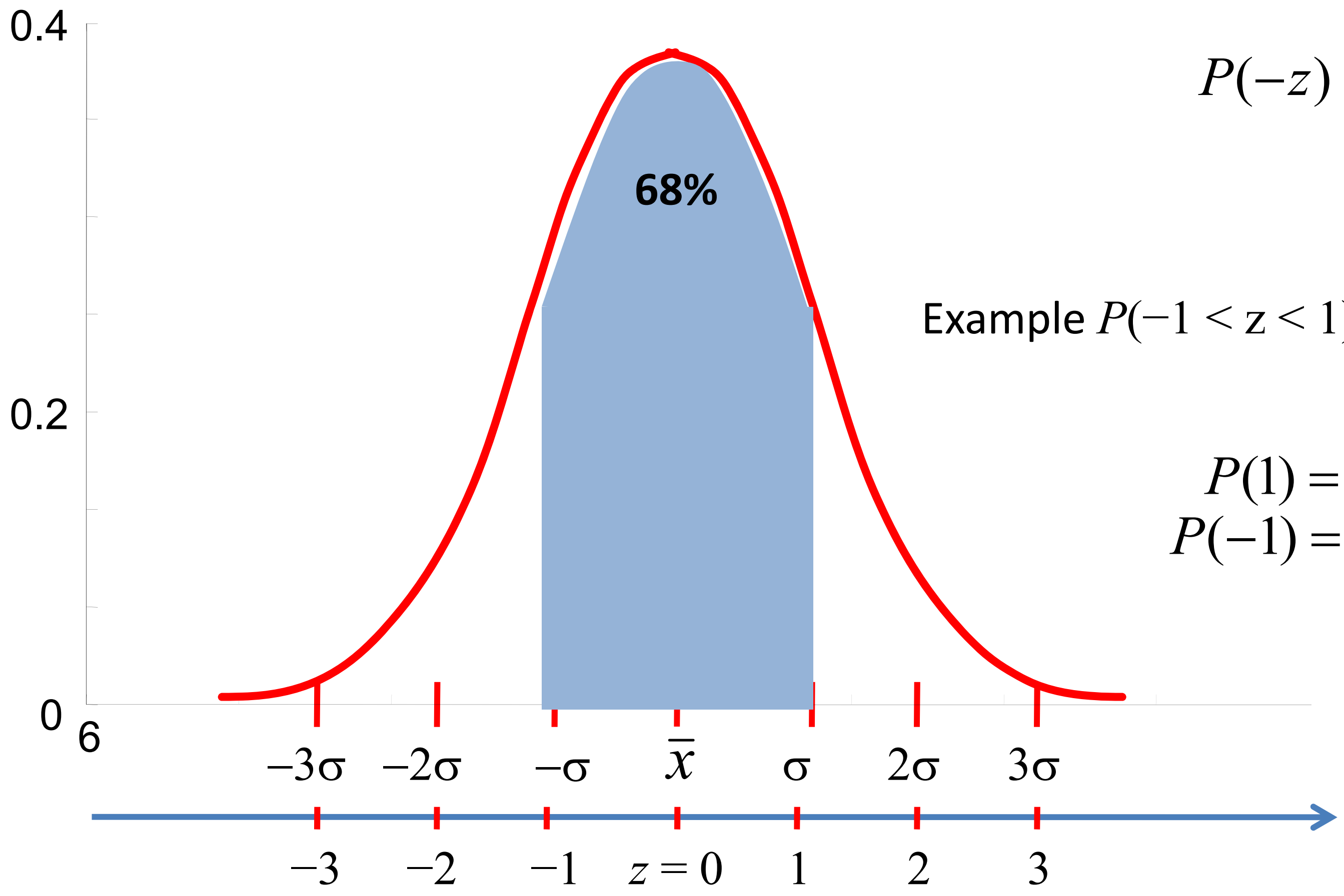
$$z = \frac{x - \bar{x}}{\sigma_x}, \sigma_z = 1$$

$$f(x) = p(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2} \longrightarrow f(z) = p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$P(-z) = 1 - P(z)$$

Example $P(-1 < z < 1) = 0.84 - 0.16 = 0.68$

$$P(1) = 0.84$$
$$P(-1) = 0.16$$



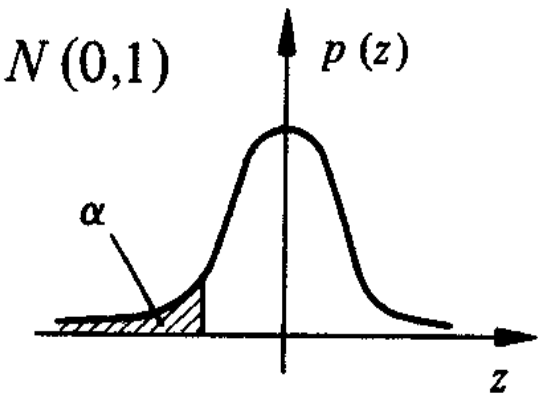
$$z = \frac{x - \bar{x}}{\sigma_x}$$

Interactive Example

- http://www.mathsisfun.com/data/standard-normal-distribution-table.html

11.1.1 Table de la distribution normale standard $N\,(0,1)$

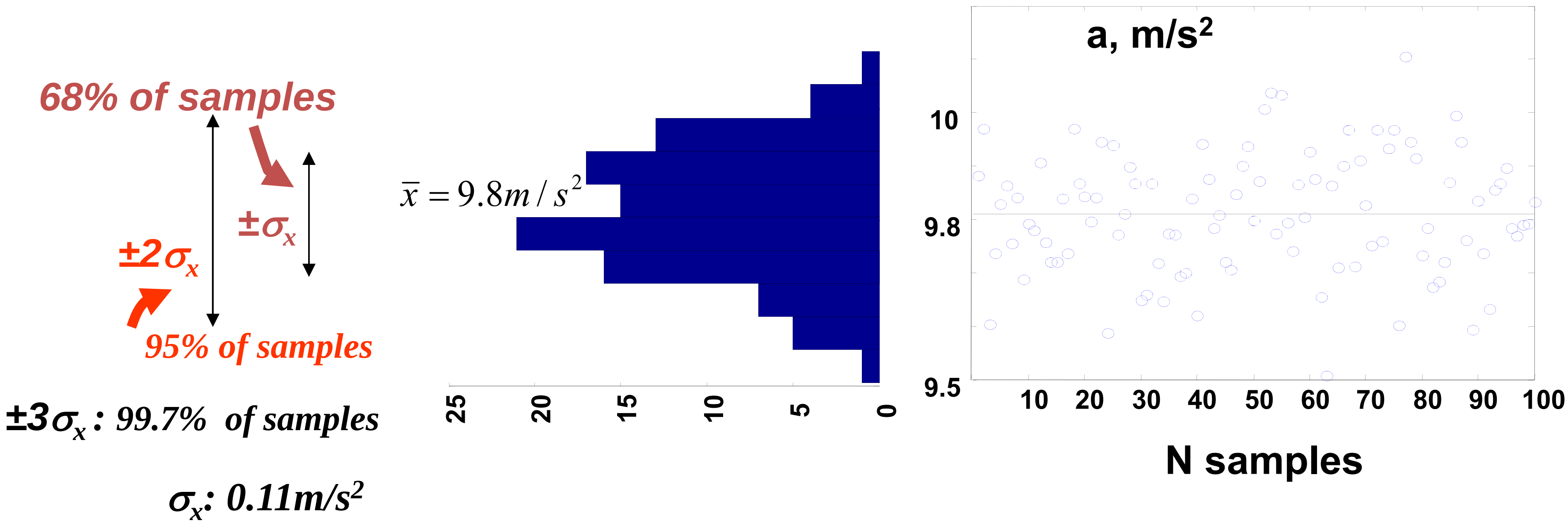
TE, Ph.Robert



$$p\,(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right)$$

z	α	z	α	z	α	z	α
−3,25	,0006	−1,00	,1587	1,05	,8531	−4,265	,00001
−3,20	,0007	− ,95	,1711	1,10	,8643	−3,719	,0001
−3,15	,0008	− ,90	,1841	1,15	,8749	−3,090	,001
−3,10	,0010	− ,85	,1977	1,20	,8849	−2,576	,005
−3,05	,0011	− ,80	,2119	1,25	,8944	−2,326	,01
−3,00	,0013	− ,75	,2266	1,30	,9032	−2,054	,02
−2,95	,0016	− ,70	,2420	1,35	,9115	−1,960	,025
−2,90	,0019	− ,65	,2578	1,40	,9192	−1,881	,03
−2,85	,0022	− ,60	,2743	1,45	,9265	−1,751	,04
−2,80	,0026	− ,55	,2912	1,50	9332	−1,645	,05
−2,75	,0030	− ,50	,3085	1,55	,9394	−1,555	,06
−2,70	,0035	− ,45	,3264	1,60	,9452	−1,476	,07
−2,65	,0040	− ,40	,3446	1,65	,9505	−1,405	,08
−2,60	,0047	− ,35	,3632	1,70	,9554	−1,341	,09
−2,55	,0054	− ,30	,3821	1,75	,9599	−1,282	,10

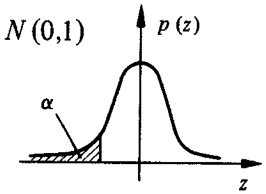
Error and its probability



Incertitude = σ with a confidence level of 68 %
Incertitude = 2σ with a confidence level of 95 %
Incertitude = 3σ with a confidence level of 99 %

11.1.1 Table de la distribution normale standard $N\,(0,1)$

$$p\,(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$$



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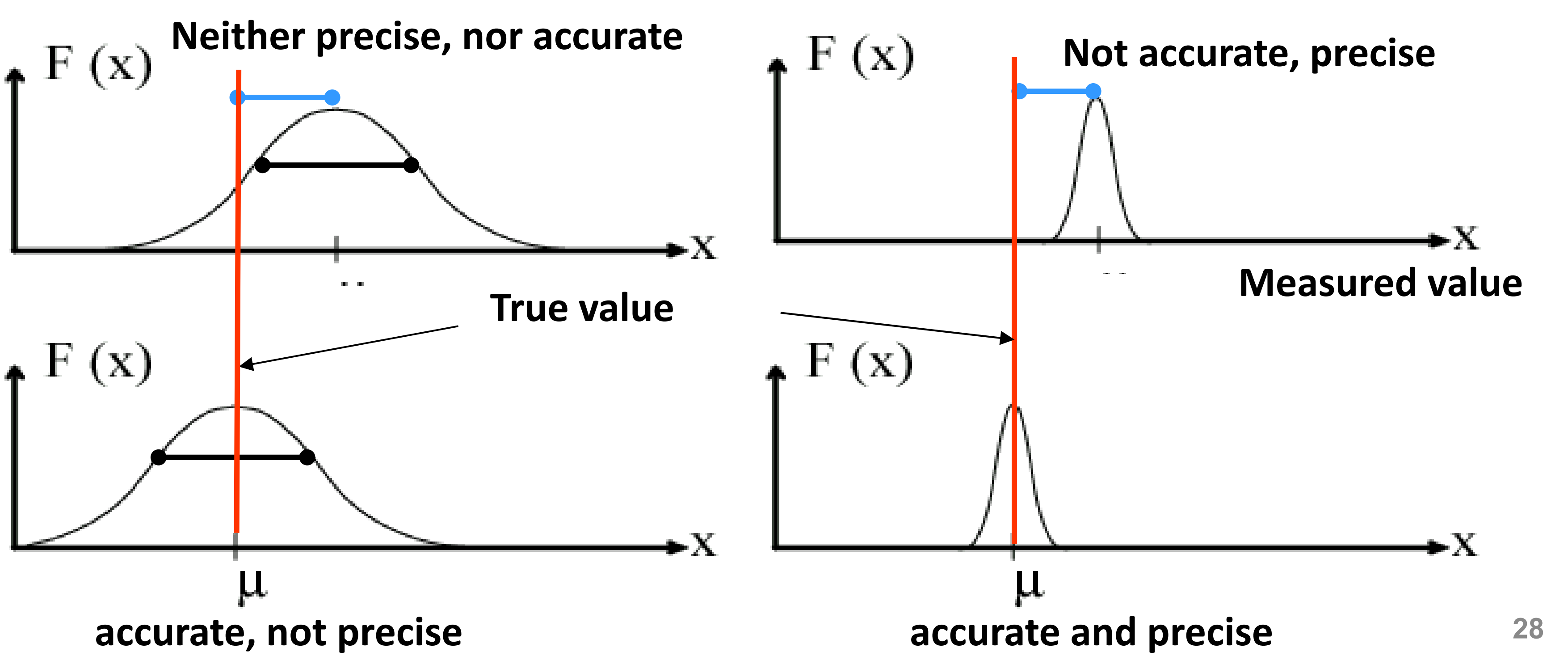
−2,65	,0040	−,40	,3446	1,65	,9505	−1,405	,08
−2,60	,0047	−,35	,3632	1,70	,9554	−1,341	,09
−2,55	,0054	−,30	,3821	1,75	,9599	−1,282	,10
−2,50	,0062	−,25	,4013	1,80	,9641	−1,036	,15
−2,45	,0071	−,20	,4207	1,85	,9678	−,842	,20
−2,40	,0082	−,15	,4404	1,90	,9713	−,674	,25
−2,35	,0094	−,10	,4602	1,95	,9744	−,524	,30
−2,30	,0107	−,05	,4801	2,00	,9772	−,385	,35
−2,25	,0122			2,05	,9798	−,253	,40
−2,20	,0139			2,10	,9821	−,126	,45
−2,15	,0158	,00	,5000	2,15	,9842	0	,50
−2,10	,0179			2,20	,9861	,126	,55
−2,05	,0202			2,25	,9878	,253	,60
−2,00	,0228	,05	,5199	2,30	,9893	,385	,65
−1,95	,0256	,10	,5398	2,35	,9906	,524	,70
−1,90	,0287	,15	,5596	2,40	,9918	,674	,75
−1,85	,0322	,20	,5793	2,45	,9929	,842	,80
−1,80	,0359	,25	,5987	2,50	,9938	1,036	,85
−1,75	,0401	,30	,6179	2,55	,9946	1,282	,90
−1,70	,0446	,35	,6368	2,60	,9953	1,341	,91
−1,65	,0495	,40	,6554	2,65	,9960	1,405	,92
−1,60	,0548	,45	,6736	2,70	,9965	1,476	,93
−1,55	,0606	,50	,6915	2,75	,9970	1,555	,94
−1,50	,0668	,55	,7088	2,80	,9974	1,645	,95
−1,45	,0735	,60	,7257	2,85	,9978	1,751	,96
−1,40	,0808	,65	,7422	2,90	,9981	1,881	,97
−1,35	,0885	,70	,7580	2,95	,9984	1,960	,975
−1,30	,0968	,75	,7734	3,00	,9987	2,054	,98
−1,25	,1056	,80	,7881	3,05	,9989	2,326	,99
−1,20	,1151	,85	,8023	3,10	,9990	2,576	,995
−1,15	,1251	,90	,8159	3,15	,9992	3,090	,999
−1,10	,1357	,95	,8289	3,20	,9993	3,719	,9999
−1,05	,1469	1,00	,8413	3,25	,9994	4,265	,99999

Precision and accuracy

- **Precision** (also reproducibility, repeatability) – degree to which repeated measurements under unchanged conditions show the same results
- **Accuracy** – closeness of the measurement result to the true value

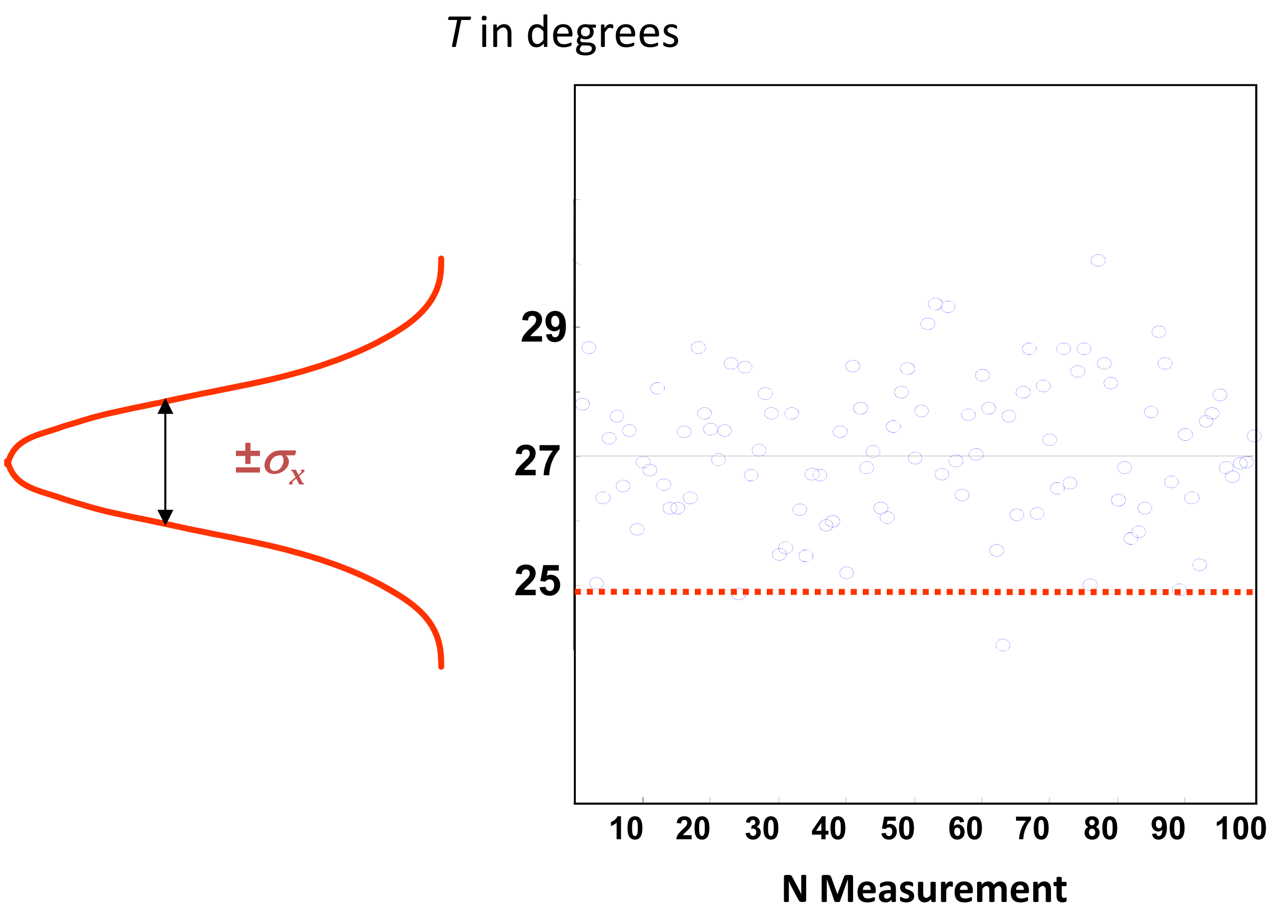


Estimation of systematic and random error

- Systematic error:
  - estimate the central tendency: average, median
  - compare to a reference system
  - hypothesis test (more on this in chapter 6)

- Random error:
  - Estimation of the dispersion: standard deviation
  - Estimation of the probability of the error

Example: temperature measurement in a room



$\bar{x} =$ $T_{average}$=27°C

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} =$$

$$= 0.9\ ^oC$$

***T* in the room μ= 25°C (known)**

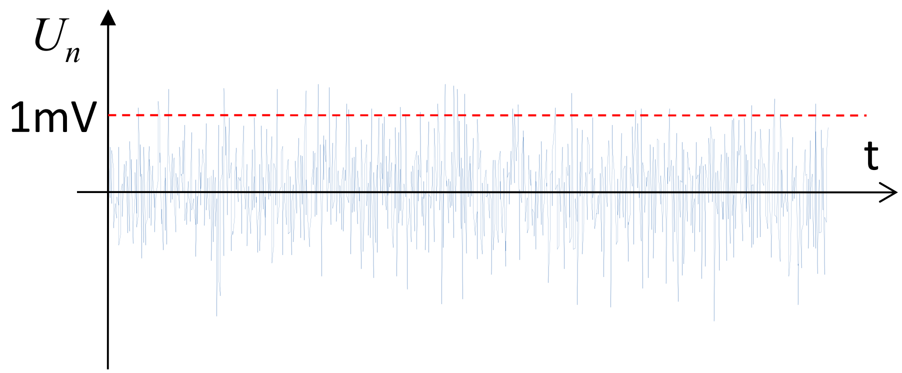
Systematic error: 2°C

Random error: ± σ_x = ± 0.9°C (p=0.68) ← standard

± $2\sigma_x$ = ± 1.8°C (p=0.95)

± $3\sigma_x$ = ± 2.7°C (p=0.99)

Example



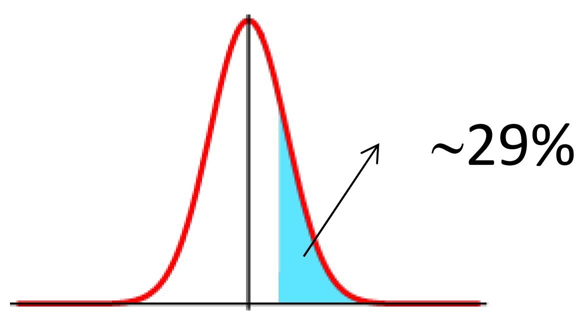
Let us consider Johnson noise from a 20 MΩ resistor at room temperature measured in a bandwidth of 10 MHz. What is the probability of measuring a voltage above 1mV at any given time at room temperature?

$$\overline{u_n^2} = 4k_B TR$$ Power spectral density of voltage variance

$$\overline{U_n^2} = 4k_B TR \cdot B = 3.23 \times 10^{-5}\ V^2$$

$$\sqrt{\overline{U_n^2}} = 1.79\ mV = \sigma$$

$$z = \frac{x-\mu}{\sigma} = \frac{1mV - 0}{1.79mV} = 0.56$$



$z = 0.56$

Increasing the precision

Example: measurement of gravity using an accelerometer

First approach:

- Acquire 500 points
- Calculate average and s.dev.:

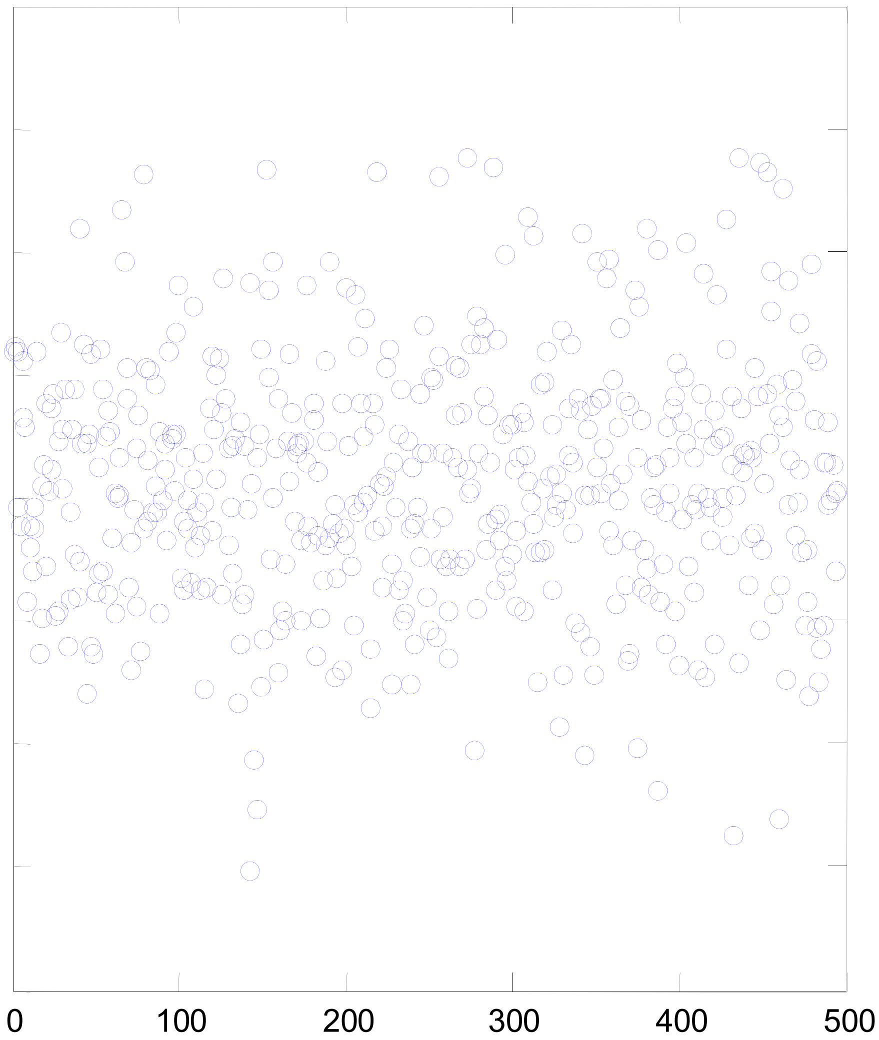
Individual measurement:

$$a = \overline{a} \pm \sigma_a \quad (p = 0.68)$$

Error for an individual measurement

$$\sigma_a = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(a_i - \overline{a})^2} = 0.1 m/s^2$$

$$a = 9.81\ m/s^2 \pm 0.1 m/s^2 \ (p = 0.68)$$



Increasing the precision

Example: measurement of gravity using an accelerometer

Second approach:

- Take those same 500 points
- During acquisition or analysis, bunch the points in groups of for example 40 points
- Calculate the averages for every group separately -> $\overline{a_{N=40}}$
- Calculate the average of group averages

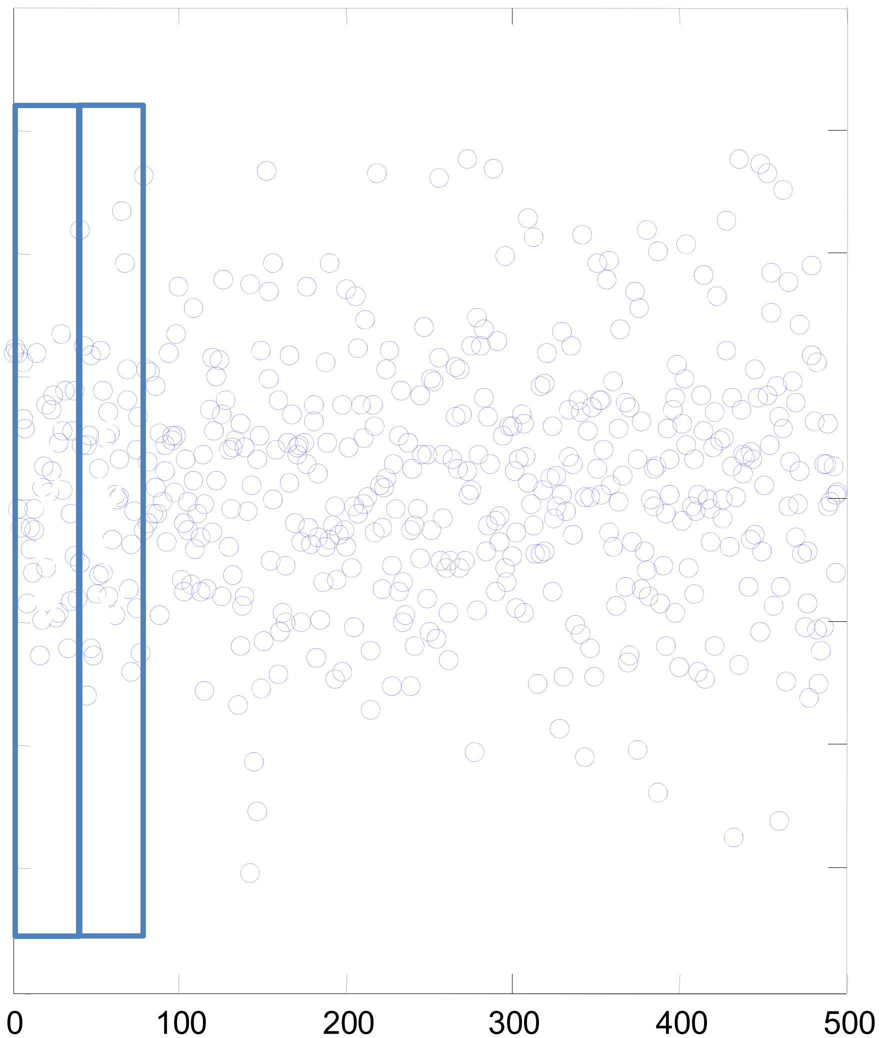
$$\overline{\overline{a_N}} = \bar{a}$$

$$\sigma_{\overline{a_N}} = \frac{\sigma}{\sqrt{N}} = \frac{0.1}{\sqrt{40}} = 0.016 m/s^2$$

Error of the «local» average

$$\overline{a_N} = \overline{\overline{a_N}} \pm \sigma_{\overline{a_N}} = \bar{a} \pm \frac{\sigma_a}{\sqrt{N}}$$

«Local» average



Estimation of the local average

- Calculating the “local” average allows us to reduce the random error and increase the precision of the measurement
- Compare $Var(x)$ and $Var(\overline{x})$
- N – number of points in a subset, M - number of subsets, $N\times M$ – total number of points

$$Var(\overline{x}) = \frac{1}{M}\sum_{j=1}^{M}\left(\overline{x}_j - \overline{\overline{x}}\right)^2, \overline{\overline{x}} = \frac{1}{M\cdot N}\sum_{k=1}^{N\cdot M}\overline{x}_k = \overline{x}$$

$$Var(\overline{x}) = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{1}{N}\sum_{i=1}^{N}x_{ji} - \overline{x}\right)^2 = \frac{1}{M}\frac{1}{N^2}\sum_j\left(\sum_i x_{ji} - N\overline{x}\right)^2 = \frac{1}{M}\frac{1}{N^2}\sum_j\left(\sum_i\left(x_{ji} - \overline{x}\right)\right)^2$$

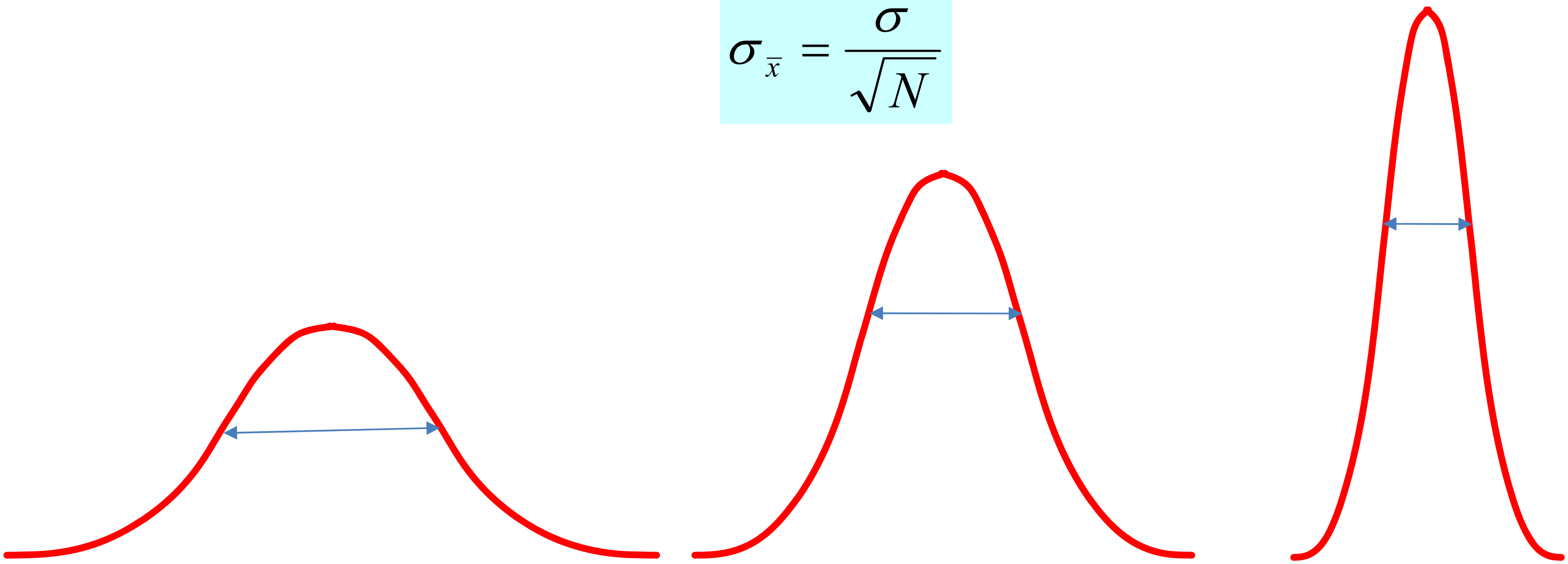
$$= \frac{1}{M}\frac{1}{N^2}\sum_j\left[\sum_i\left(x_{ji} - \overline{x}\right)^2 + \overbrace{\sum_{i\neq l}\left(x_{ji} - \overline{x}\right)\cdot\left(x_{jl} - \overline{x}\right)}^{=0\text{, because the samples are independent}}\right] = \frac{1}{M}\frac{1}{N^2}\sum_j\left[\sum_i\left(x_{ji} - \overline{x}\right)^2\right]$$

$$= \frac{1}{M}\frac{1}{N^2}\sum_k\left(x - \overline{x}\right)^2 = \frac{1}{M}\frac{1}{N^2}M\cdot N\cdot Var(x) = \frac{Var(x)}{N}$$

Precision of the average

Increasing N -> increasing precision (decreasing dispersion)

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

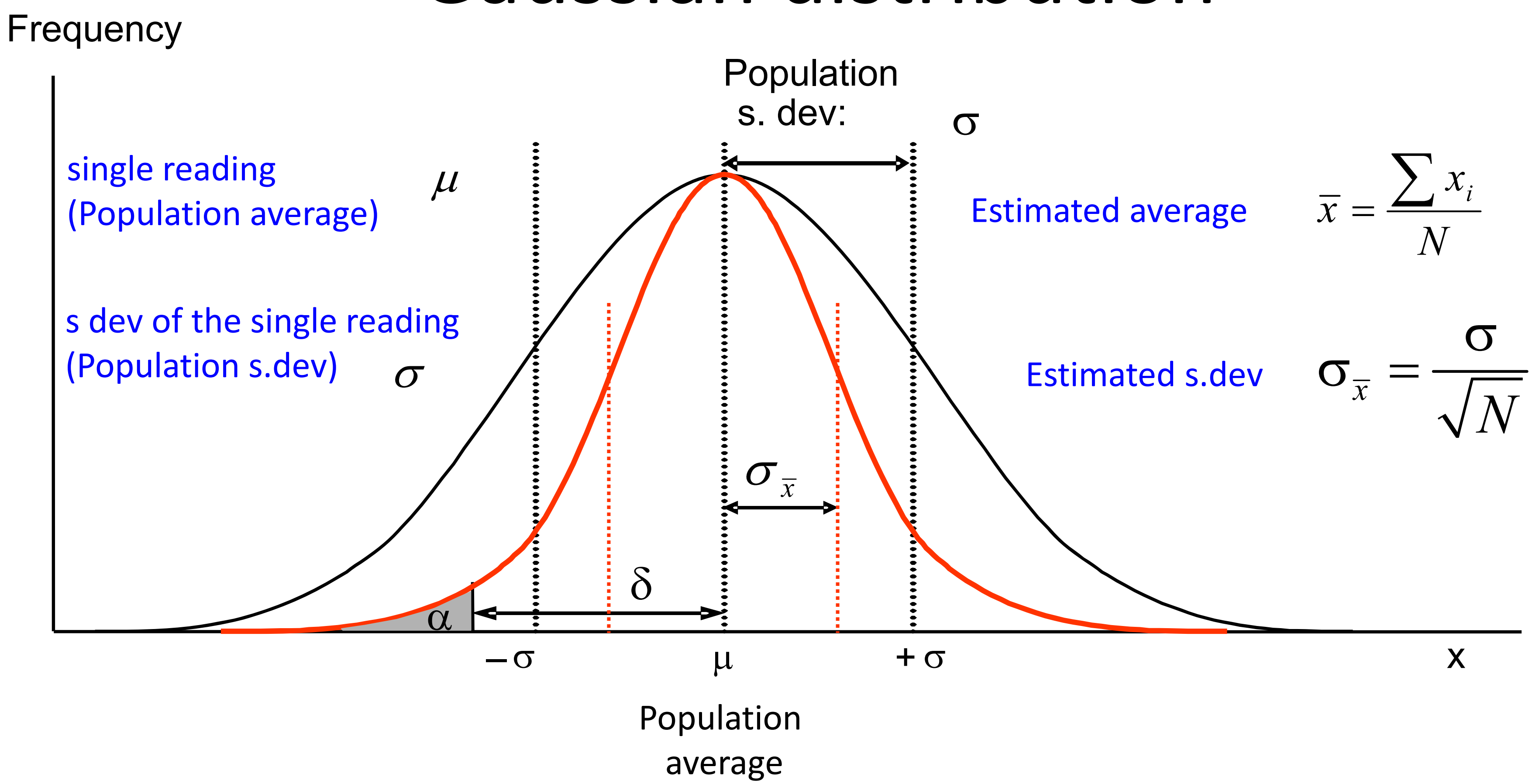


Question

In practice, you would not take 1000 points but you would have a sensor that gives a single reading with an error σ with probability of 68% (this is usually specified by the manufacturer).

How many readings $\boldsymbol{N}$ do we need in order to estimate the average value with error δ and associated probability $\boldsymbol{p}$?

Gaussian distribution



- For a normal distribution 68% of the points are between ± σ
- standard deviation of the average < standard deviation of the population (single measurement)
- Error of the estimation = standard deviation of the average
- $p = 1 - 2\alpha$ – probability that the difference between a single reading and the average of the readings is less than δ

Confidence interval of the average

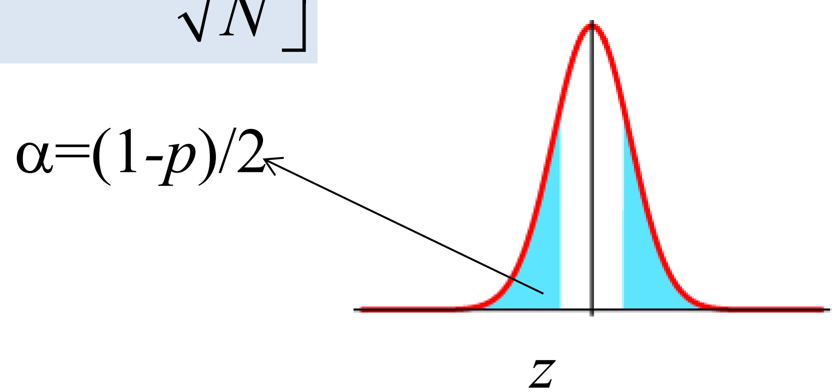
$1 \cdot \sigma_{\bar{x}}$ confidence level of 68% $CI_{68\%} = \left[\bar{x} - \frac{\sigma}{\sqrt{N}},\ \bar{x} + \frac{\sigma}{\sqrt{N}} \right]$

$2 \cdot \sigma_{\bar{x}}$ confidence level of 95% $CI_{95\%} = \left[\bar{x} - 2\frac{\sigma}{\sqrt{N}},\ \bar{x} + 2\frac{\sigma}{\sqrt{N}} \right]$

$z \cdot \sigma_{\bar{x}}$ confidence level of $p(z)$ (see in the table)

$$CI_{p\%} = \left[\bar{x} - z_{\alpha}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha}\frac{\sigma}{\sqrt{N}} \right]$$

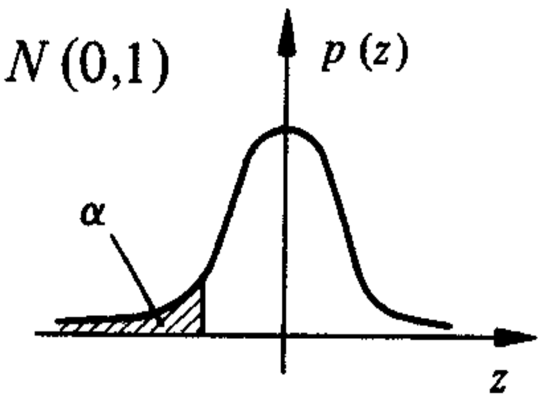
$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$$



11.1.1 Table de la distribution normale standard $N(0,1)$

TE, Ph.Robert

$$p(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$$

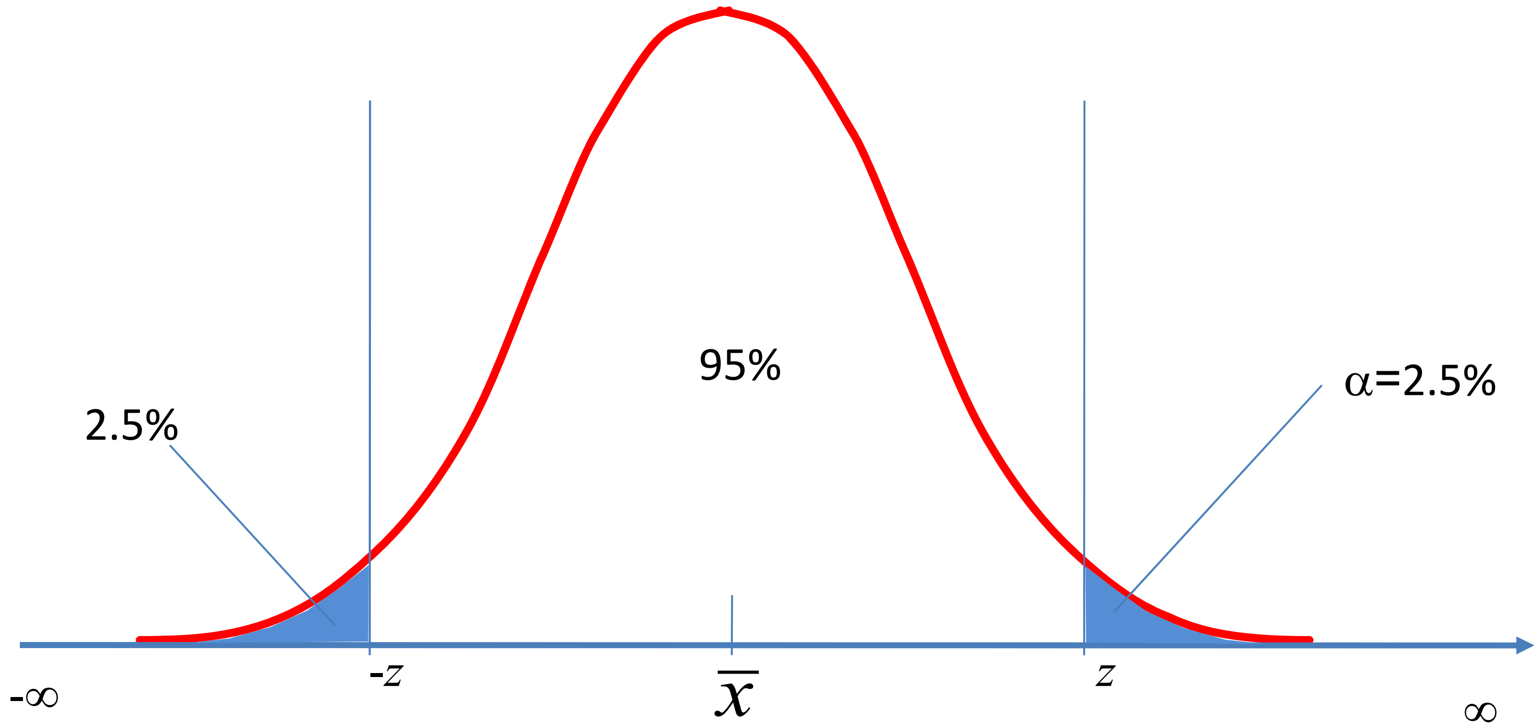


z	α	z	α	z	α	z	α
−3,25	,0006	−1,00	,1587	1,05	,8531	−4,265	,00001
−3,20	,0007	− ,95	,1711	1,10	,8643	−3,719	,0001
−3,15	,0008	− ,90	,1841	1,15	,8749	−3,090	,001
−3,10	,0010	− ,85	,1977	1,20	,8849	−2,576	,005
−3,05	,0011	− ,80	,2119	1,25	,8944	−2,326	,01
−3,00	,0013	− ,75	,2266	1,30	,9032	−2,054	,02
−2,95	,0016	− ,70	,2420	1,35	,9115	−1,960	,025
−2,90	,0019	− ,65	,2578	1,40	,9192	−1,881	,03
−2,85	,0022	− ,60	,2743	1,45	,9265	−1,751	,04
−2,80	,0026	− ,55	,2912	1,50	9332	−1,645	,05
−2,75	,0030	− ,50	,3085	1,55	,9394	−1,555	,06
−2,70	,0035	− ,45	,3264	1,60	,9452	−1,476	,07
−2,65	,0040	− ,40	,3446	1,65	,9505	−1,405	,08
−2,60	,0047	− ,35	,3632	1,70	,9554	−1,341	,09
−2,55	,0054	− ,30	,3821	1,75	,9599	−1,282	,10

Example: error of a gyroscope

- In order to estimate the error of a gyroscope reading, we carry out 36 identical tests. They consist of turning the gyroscope by 360°, taking the angular velocity readings while the gyroscope is turning and then integrating the velocity (which should give us a total angle of 360°)
- We get $\bar{x} = 359.8\,\mathrm{deg};\ \sigma = 2.4\,\mathrm{deg}$

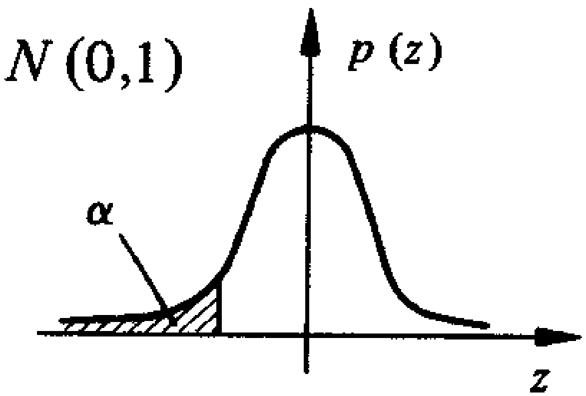
- Calculate the 95% confidence interval for this test (the interval in which the value of the next test with 36 readings would be, with a 95% probability)



$$\bar{x} \pm z_{2.5\%} \frac{\sigma}{\sqrt{N}} = 359.8 \pm 1.96 \frac{2.4}{\sqrt{36}} = 359.8 \pm 0.8\,\text{deg}$$

There is a probability of 95% of getting the next average between 359.0 and 360.6 deg (358.8 ± 0.8)

11.1.1 Table de la distribution normale standard $N\,(0,1)$



$$p\,(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$$

z	α	z	α	z	α	z	α
−3,25	,0006	−1,00	,1587	1,05	,8531	−4,265	,00001
−3,20	,0007	− ,95	,1711	1,10	,8643	−3,719	,0001
−3,15	,0008	− ,90	,1841	1,15	,8749	−3,090	,001
−3,10	,0010	− ,85	,1977	1,20	,8849	−2,576	,005
−3,05	,0011	− ,80	,2119	1,25	,8944	−2,326	,01
−3,00	,0013	− ,75	,2266	1,30	,9032	−2,054	,02
−2,95	,0016	− ,70	,2420	1,35	,9115	−1,960	,025
−2,90	,0019	− ,65	,2578	1,40	,9192	−1,881	,03
−2,85	,0022	− ,60	,2743	1,45	,9265	−1,751	,04
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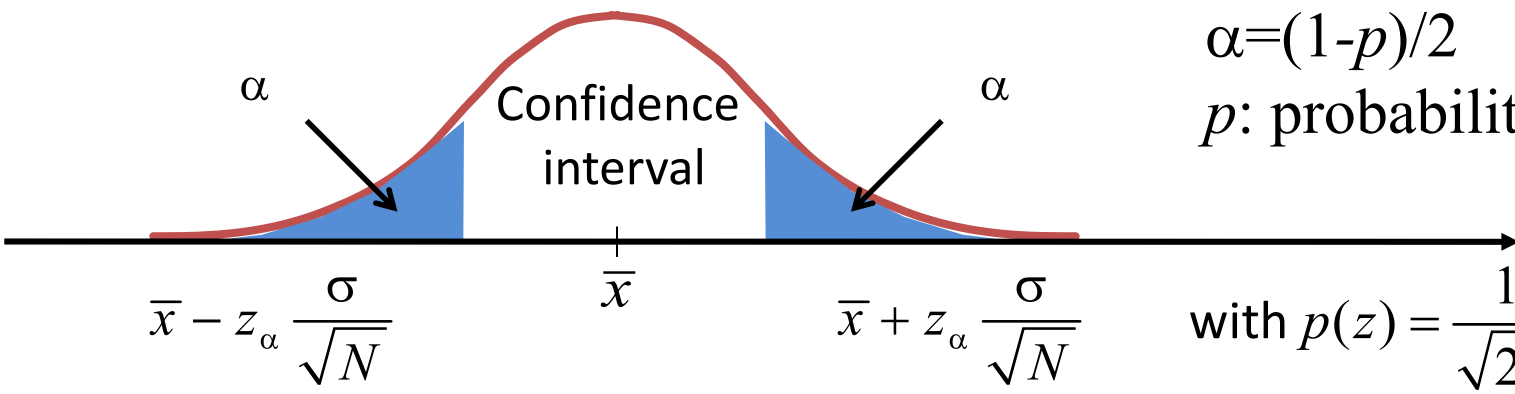
−2,65	,0040	− ,40	,3446	1,65	,9505	−1,405	,08
−2,60	,0047	− ,35	,3632	1,70	,9554	−1,341	,09
−2,55	,0054	− ,30	,3821	1,75	,9599	−1,282	,10

Normal distribution of the average

- If the standard deviation of an individual measurement is **known** and if the total number of measurements N **> 30**, we use the normal distribution

- Confidence interval: $$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{N}} < \mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{N}}$$



$\alpha = (1-p)/2$

p: probability

with $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

Student distribution of the average

- Proposed by William Sealy Gosset in 1908
- If the standard deviation of an individual measurement is **estimated** or if the total number of measurements N < 30, we use the Student distribution
- Confidence interval

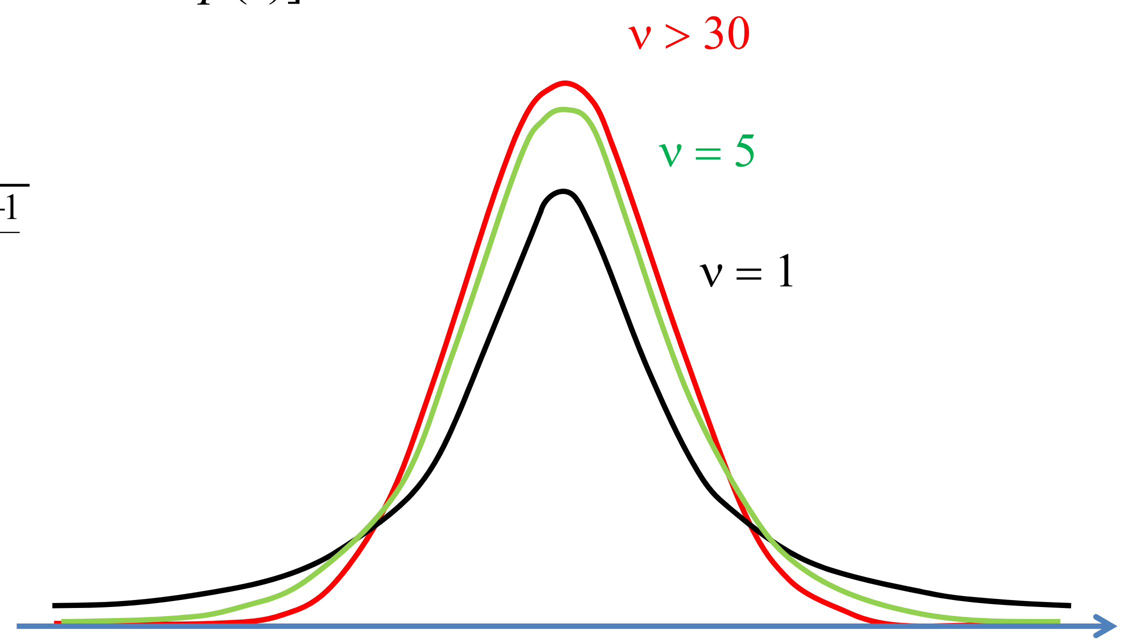
$$\overline{x} - t_\alpha \frac{\sigma}{\sqrt{N}} < \mu < \overline{x} + t_\alpha \frac{\sigma}{\sqrt{N}}$$

Student distribution of the average

probability

$$p(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$

[For normal distribution, we had $p(z)$]

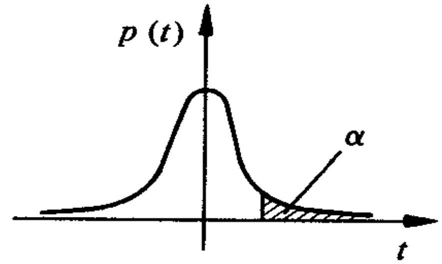


$$\Gamma(y) = \int_0^{+\infty} x^{y-1} e^{-x} dx$$

ν : number of degrees of freedom = $N - 1$ where N is the number of measurements

11.1.2 Table de la distribution T de Student

$$p(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$



ν	α									
	0,4	0,25	0,1	0,05	0,025	0,01	0,005	0,0025	0,001	0,0005
1	0,325	1,000	3,078	6,314	12,706	31,821	63,657	127,32	318,31	636,62
2	,289	0,816	1,886	2,920	4,303	6,965	9,925	14,089	22,327	31,598
3	,277	,765	1,638	2,353	3,182	4,541	5,841	7,453	10,214	12,924
4	,271	,741	1,533	2,132	2,776	3,747	4,604	5,598	7,173	8,610
5	0,267	0,727	1,476	2,015	2,571	3,365	4,032	4,773	5,893	6,869
6	,265	,718	1,440	1,943	2,447	3,143	3,707	4,317	5,208	5,959
7	,263	,711	1,415	1,895	2,365	2,998	3,499	4,029	4,785	5,408
8	,262	,706	1,397	1,860	2,306	2,896	3,355	3,833	4,501	5,041
9	,261	,703	1,383	1,833	2,262	2,821	3,250	3,690	4,297	4,781
10	0,260	0,700	1,372	1,812	2,228	2,764	3,169	3,581	4,144	4,587
11	,260	,697	1,363	1,796	2,201	2,718	3,106	3,497	4,025	4,437
12	,259	,695	1,356	1,782	2,179	2,681	3,055	3,428	3,930	4,318
13	,259	,694	1,350	1,771	2,160	2,650	3,012	3,372	3,852	4,221
14	,258	,692	1,345	1,761	2,145	2,624	2,977	3,326	3,787	4,140
15	0,258	0,691	1,341	1,753	2,131	2,602	2,947	3,286	3,733	4,073
16	,258	,690	1,337	1,746	2,120	2,583	2,921	3,252	3,686	4,015
17	,257	,689	1,333	1,740	2,110	2,567	2,898	3,222	3,646	3,965
18	,257	,688	1,330	1,734	2,101	2,552	2,878	3,197	3,610	3,922
19	,257	,688	1,328	1,729	2,093	2,539	2,861	3,174	3,579	3,883
20	0,257	0,687	1,325	1,725	2,086	2,528	2,845	3,153	3,552	3,850
21	,257	,686	1,323	1,721	2,080	2,518	2,831	3,135	3,527	3,819
22	,256	,686	1,321	1,717	2,074	2,508	2,819	3,119	3,505	3,792
23	,256	,685	1,319	1,714	2,069	2,500	2,807	3,104	3,485	3,767
24	,256	,685	1,318	1,711	2,064	2,492	2,797	3,091	3,467	3,745

Sample size – normal distribution

- How many points should we acquire for estimating the average with an error δ and probability p?

- Case 1: we know σ (the sdev. of a single measurement). We therefore use the normal distribution

$$\alpha = \frac{1-p}{2} \qquad \bar{x} = \mu \pm \delta = \mu \pm z_\alpha \frac{\sigma}{\sqrt{N}}$$

1. Define the error: $\delta = \bar{x} - \mu = z_\alpha \frac{\sigma}{\sqrt{N}}$

2. Find z from α (table)

3. Number of points N is then: $N = z_\alpha^2 \left(\frac{\sigma}{\delta}\right)^2$

Example

- Two sensors have noisy outputs with U_1 = 9.8 V (σ_1 = 1V) and U_2 = 9.35 V (σ_2 = 1.32 V). Calculate the smallest number of samples so that values of U_1 and U_2 are within 1% of the actual value with a confidence of 90% and 99%?

$$N = z_\alpha^2 \left(\frac{\sigma}{\delta} \right)^2$$

$z_{5\%}$ = 1.65; $z_{0.5\%}$ = 2.58;
δ_1 = 0.098V, δ_2 = 0.0935V

90%:

$N_1=1.65^2(1^2/0.098^2)=283$ $N_2=1.65^2(1.32^2/0.0935^2)=543$

99%:

$N_1=2.58^2(1^2/0.098^2)=693$ $N_2=2.58^2(1.32^2/0.0935^2)=1327$

Sample size – Student distribution

- How many points should we acquire for estimating the average with an error δ with probability p?
- Case 2: we do not know σ (the s. dev. of a single measurement or in other words we do not know the specs of the sensor)

$$\alpha = \frac{1-p}{2}$$

1. Choose an initial sample size N' by performing N' measurements

2. Estimate $\bar{x}$ and σ

3. Find t_α from the table with ν=N'-1

4. Optional: calculate error δ'(the error for this set of measurements):

$$\delta' = t_\alpha \frac{\sigma}{\sqrt{N}}$$

5. Calculate desired sample size N:

$$N = t_\alpha^2 \left(\frac{\sigma}{\delta} \right)^2$$

Example

- Based on 6 measurements, the average melting temperature of tin is estimated to be 232.26 °C with a standard deviation of 0.14°C. If we use this value as the real melting temperature, calculate the maximum error with a confidence level of 98%. How many measurements should we make to have an error of 0.05 °C?

N = 6, σ = 0.14, $t_{0.01}$ = 3.365 (for $\nu = N - 1 = 5$)

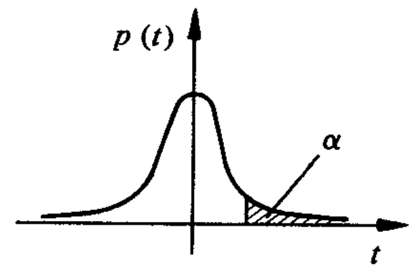
$$\delta = t_{0.01}\left(\frac{\sigma}{\sqrt{N}}\right) = 3.365\left(\frac{0.14}{\sqrt{6}}\right) = 0.19^{\circ}C$$

Error = 0.19°C (p=98%)

$$N = 3.365^2\left(\frac{0.14}{0.05}\right)^2 = 89$$

11.1.2 Table de la distribution T de Student

$$p(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$



ν	α									
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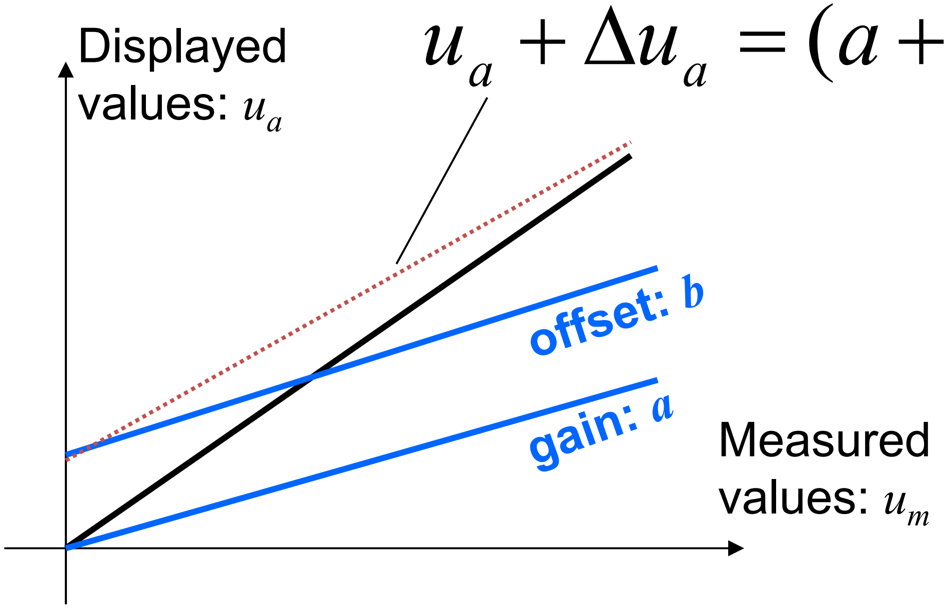
24	,256	,685	1,318	1,711	2,064	2,492	2,797	3,091	3,467	3,745

$\nu = n-1$

Precision of digital instruments

$$u_a = a \cdot u_m + b$$

$$u_a + \Delta u_a = (a + \Delta a) u_m + b + \Delta b$$



$$\Delta u_a = \Delta a \cdot u_m + \Delta b$$

Error: ±(%reading + n digit)

Error on the gain

Error on the offset

Precision of digital instruments

- **Error: ± (% reading + n digit)** -
  - for example BBC M2030: ± (0.1%reading+1d)

For displayed value of 4.00 V:

1d : 0.01 V

0.1%reading = 0.1% × 4 V = 4 mV

Error = ±(4 mV + 0.01 V) = ±0.014 V

Relative error = ±0.014/4 = 0.35%

Precision of digital instruments

- **Error: ± (%reading + %FS)**
  - for example Philips 2514 voltmeter: ± (0.1%reading + 0.02%FS)

For displayed value of 4.000 V:

0.02%FS (= 10 V) = 0.002 V

Error: ±(0.1% × 4 V + 0.002 V) = ±0.006 V

Relative error: ±0.006 V / 4 V = 0.15%

Key points

- Determining the precision of a measurement and its probability from the systematic and random errors
- Estimation of the systematic error from the central tendency and the random error from the dispersion
- Using normal and Student distributions for modelling the dispersion
- Calculating the error on the average
- Estimating the confidence interval for an average
- Estimating the sample size