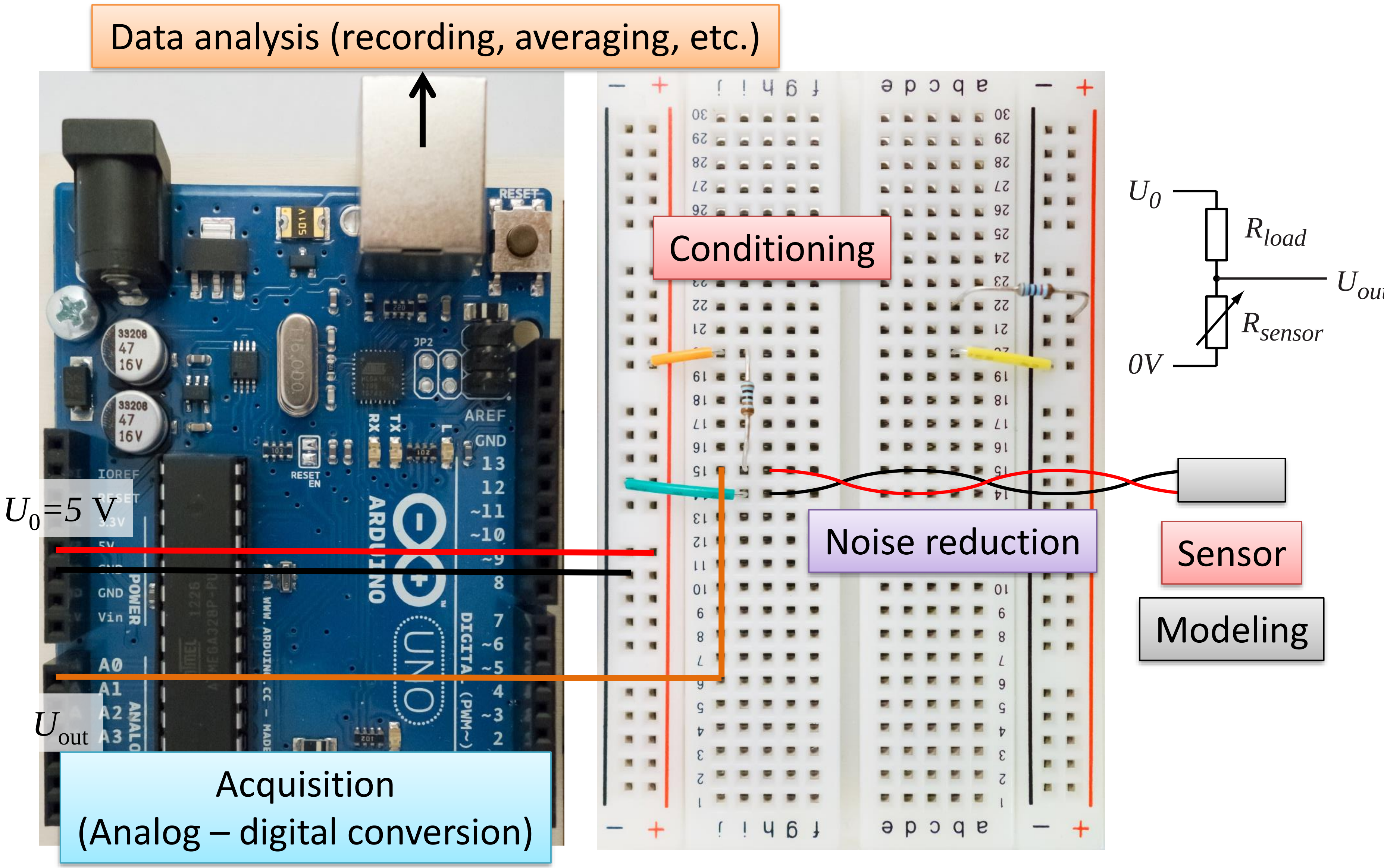
Measurement systems

Lecturer: Andras Kis

Chapter 6: Comparison

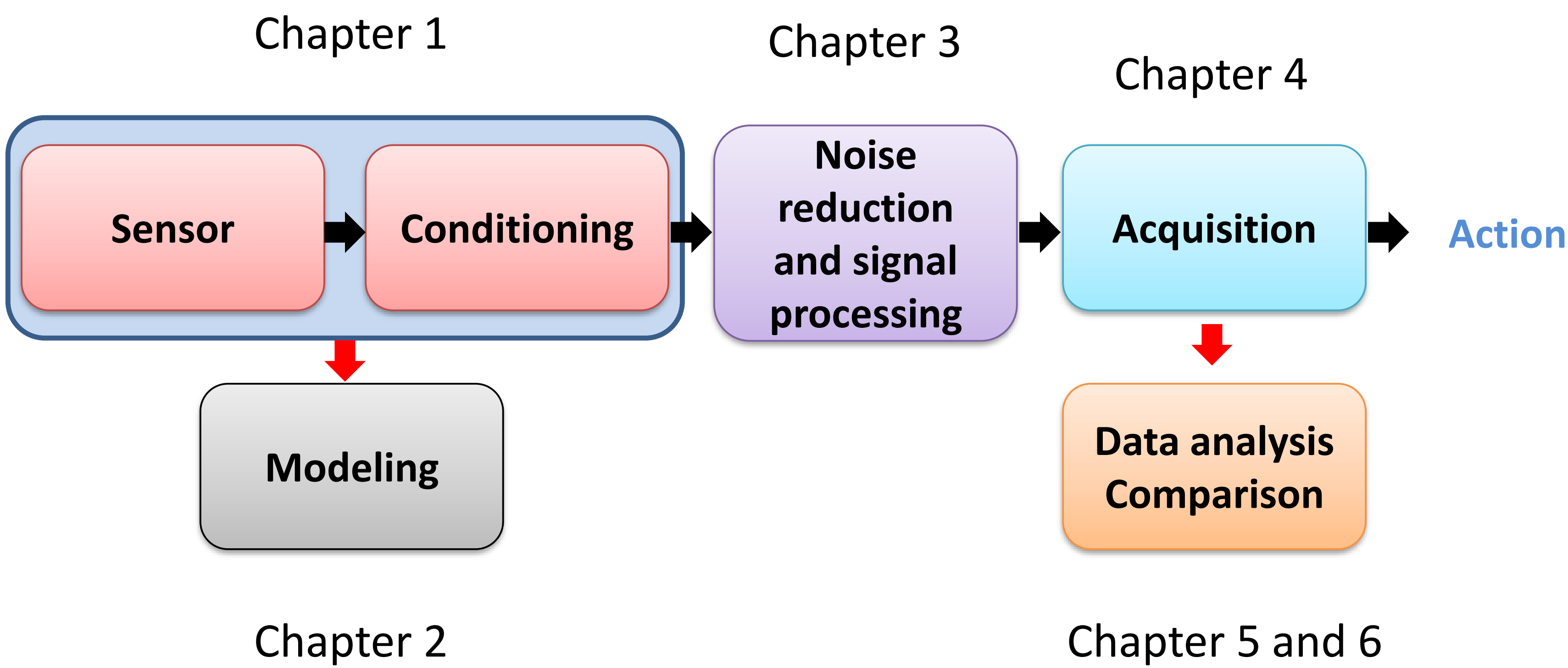
Measurement chain



Arduino UNO board

Conditioning circuit

Measurement chain



Chapter 6: Comparing measurement results

- Dispersion diagram
- Regression and correlation
- Hypothesis testing

Examples of questions to answer

- Metrology

  Q1: does the average value supplied by the sensor correspond to the actual (real) value we are trying to measure?

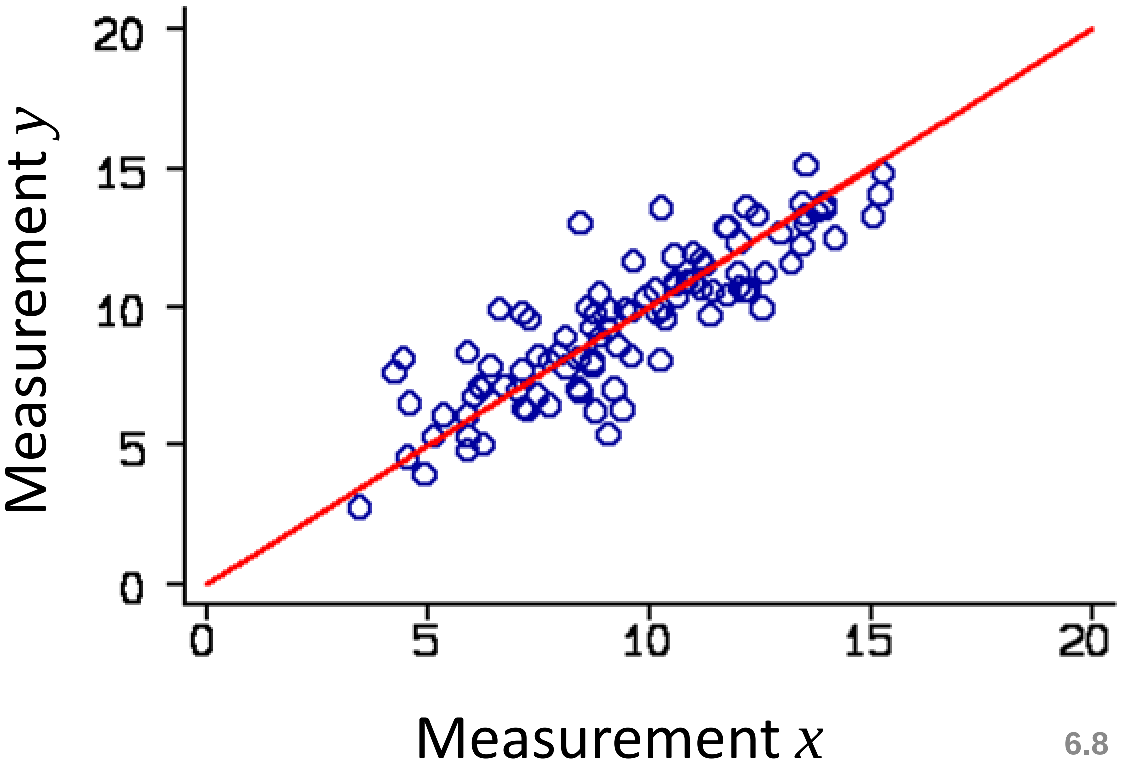
  Q2: which one of two or more measurement methods is more precise (smaller σ) / correct (closer to the real value)?

  Q3: which of the two or more noise reduction methods is more efficient (results in a smaller σ)?

- Other domains
  - effectiveness of a medical treatment
  - differences between populations

Dispersion diagram

- Presentation of (x,y) coordinate pairs
- Highlighting a relationship
- Statistical distribution?
- Total number of measurements?



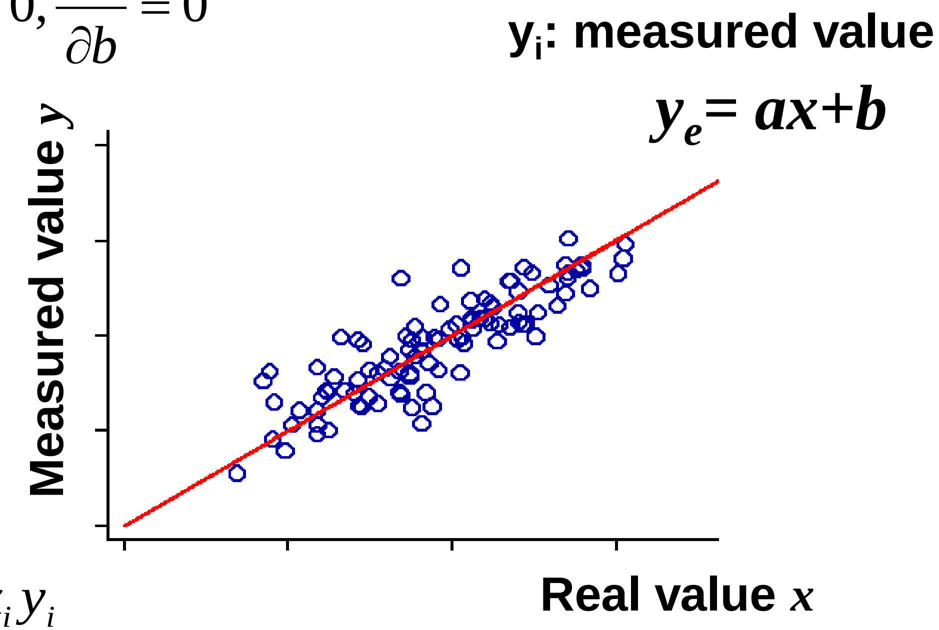
Regression and correlation

- Identifying a linear relationship between x and y
- Linear regression line $y_e = ax + b$

Minimize $D = \sum_{i=1}^{N} (y_i - y_{ei})^2$ $\quad \frac{\partial D}{\partial a} = 0, \frac{\partial D}{\partial b} = 0$

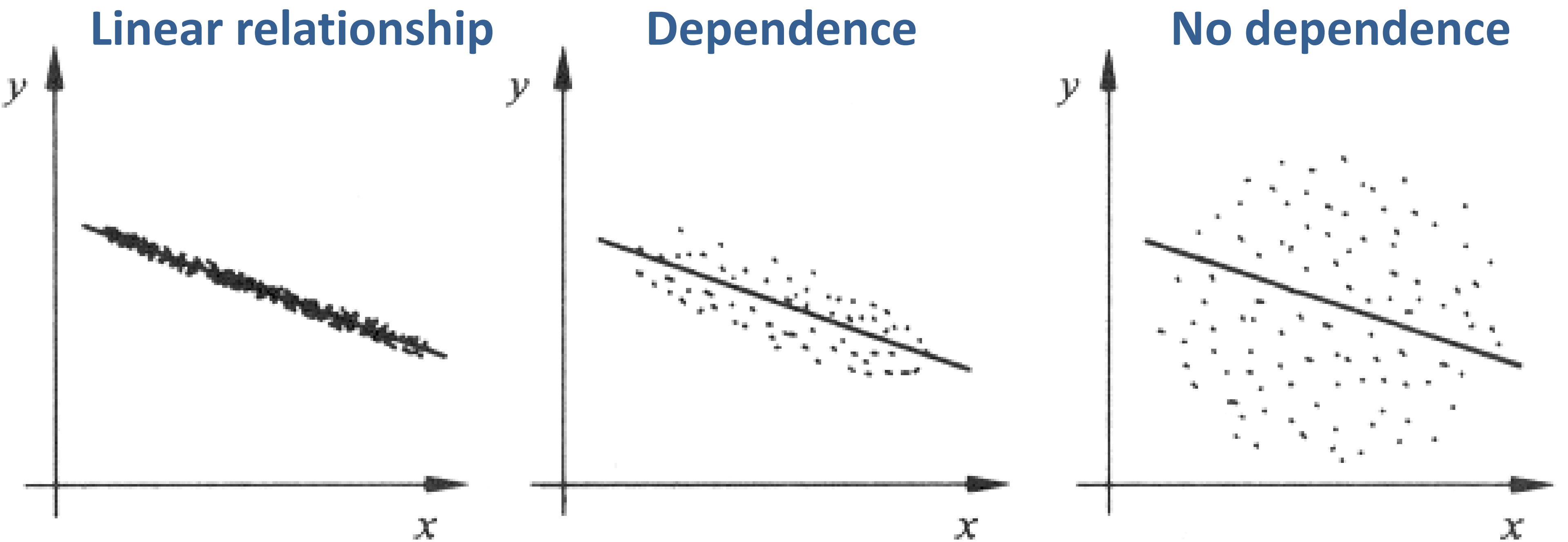
$$a = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N}\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2}$$

$$b = \bar{y} - a\bar{x} = \frac{\frac{1}{N}\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \frac{1}{N}\sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2}$$



Regression and correlation

- All these data sets result in the same regression line:



- How do we measure the significance of the regression line, how faithfully it represents the original data?

Correlation coefficient

- R measures the strength of the linear relationship

$$R = \frac{s^2{}_{xy}}{s_x s_y} = a\,\frac{s_x}{s_y} \qquad s_{xy}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

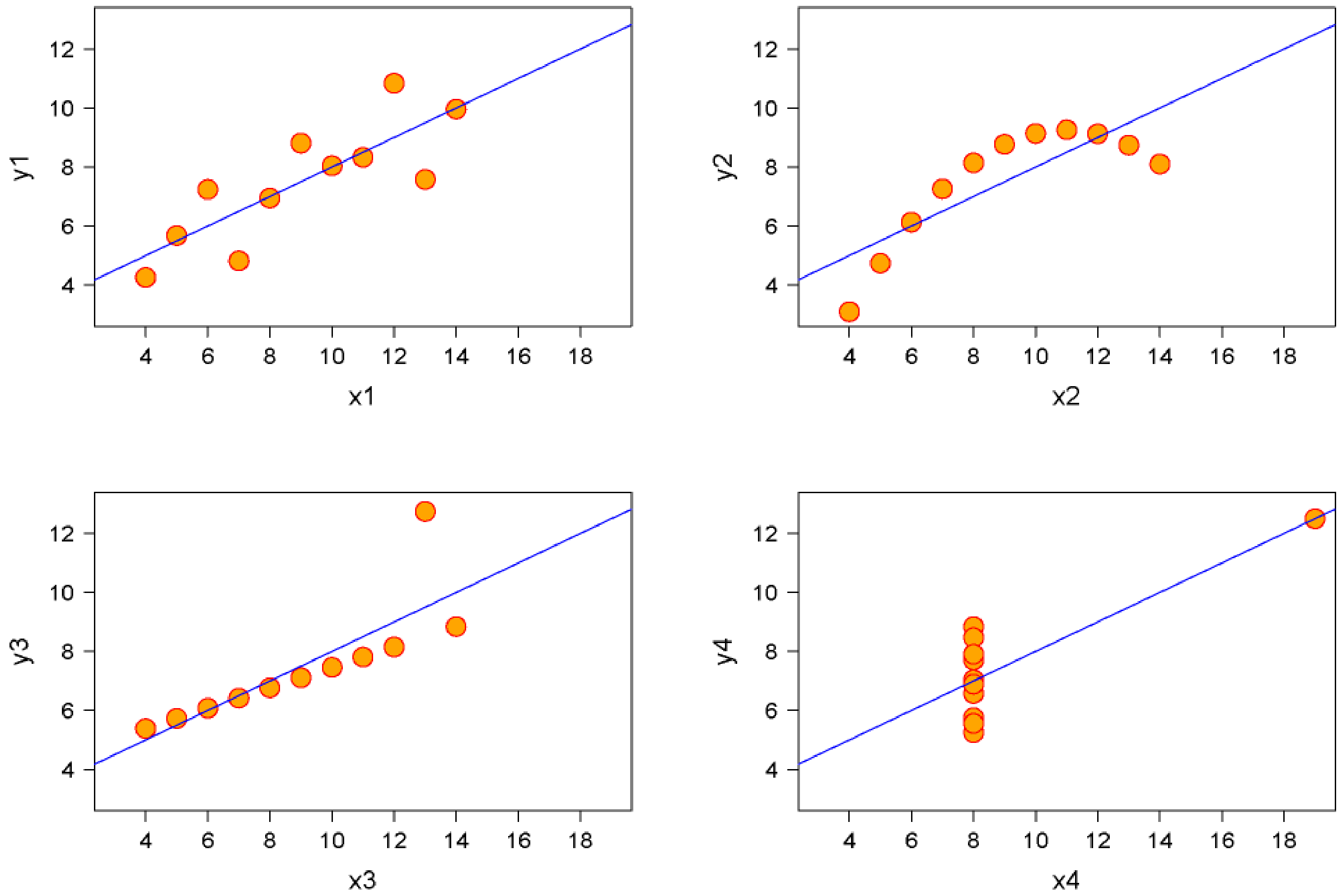
$$-1 < R < 1 \qquad s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \qquad s_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$s^2{}_x$: estimates the variance of x ($\sigma_x{}^2$)
$s^2{}_y$: estimates the variance of y ($\sigma_y{}^2$)
$s^2{}_{xy}$: estimates the covariance of x and y

- R does not measure the strength of a non-linear relationship!

Examples of correlation

N=11, R=0.81



Source: wikipedia

Correlation and causation

Correlation does not mean causation: correlation is necessary but not sufficient

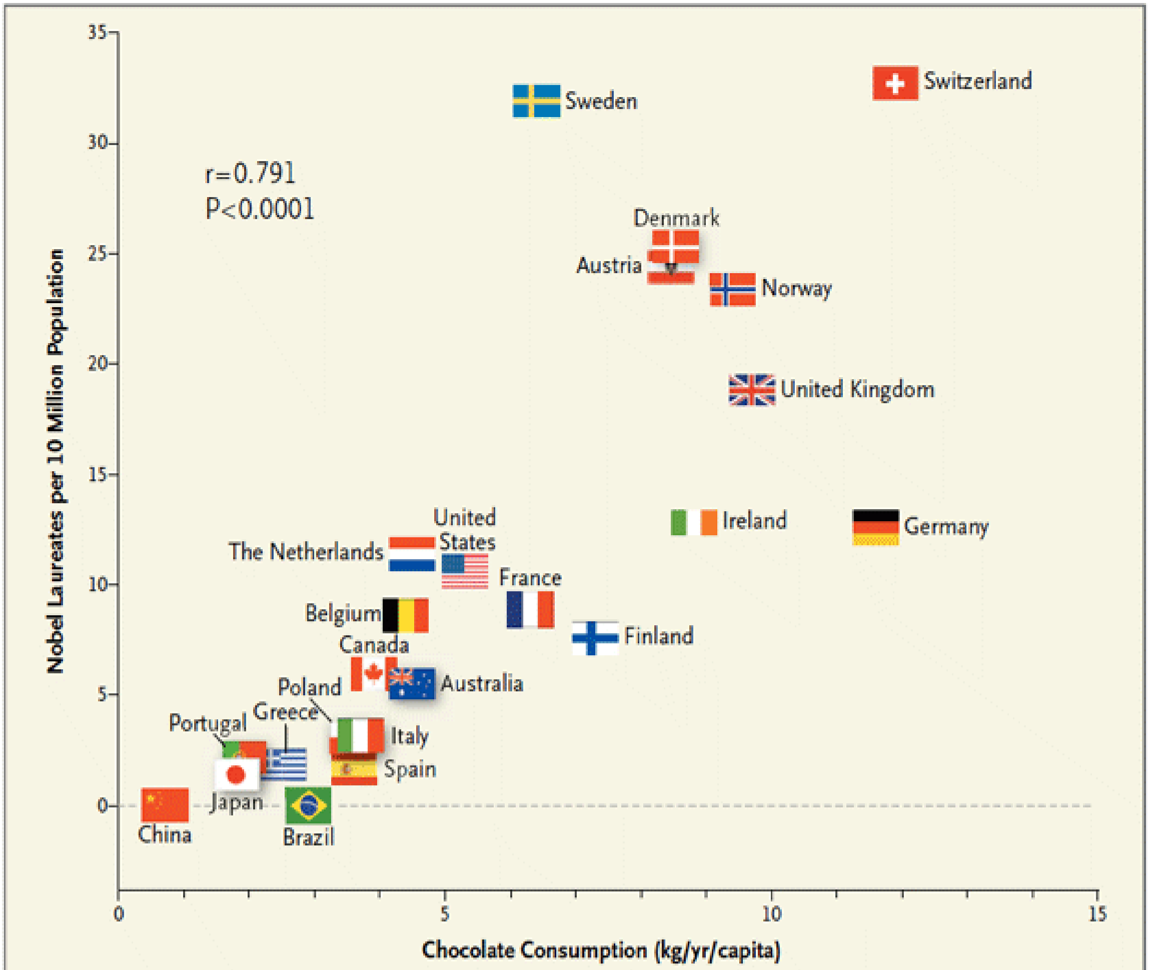


Figure 1. Correlation between Countries' Annual Per Capita Chocolate Consumption and the Number of Nobel Laureates per 10 Million Population.

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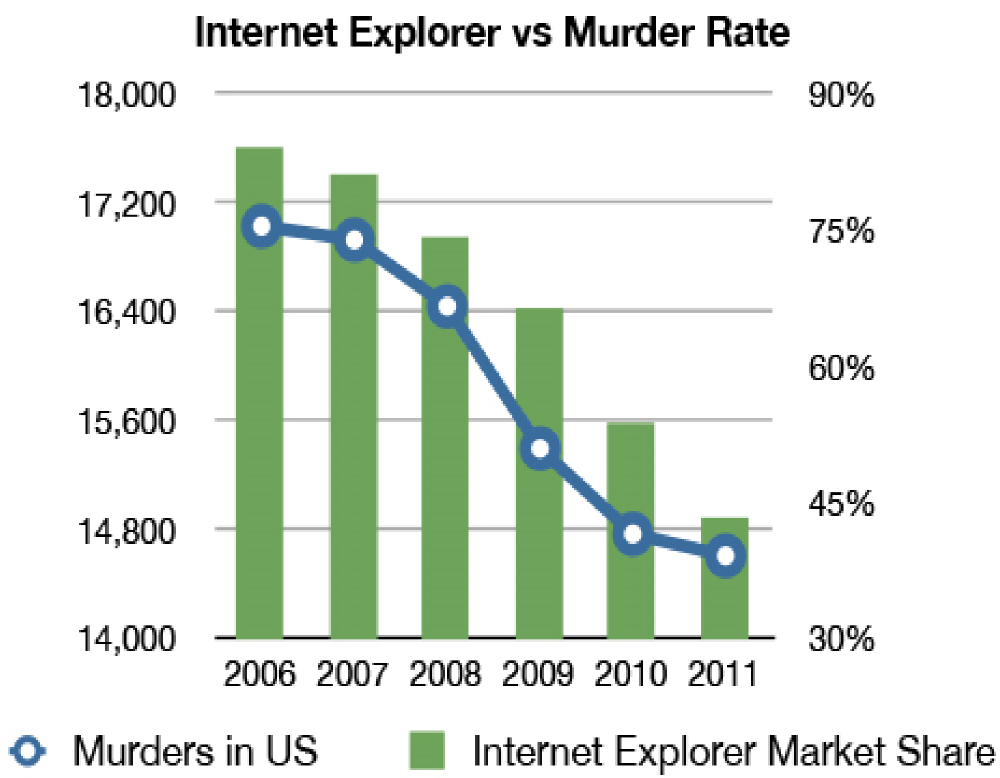
Correlation and causation



Source: [Gizmodo]

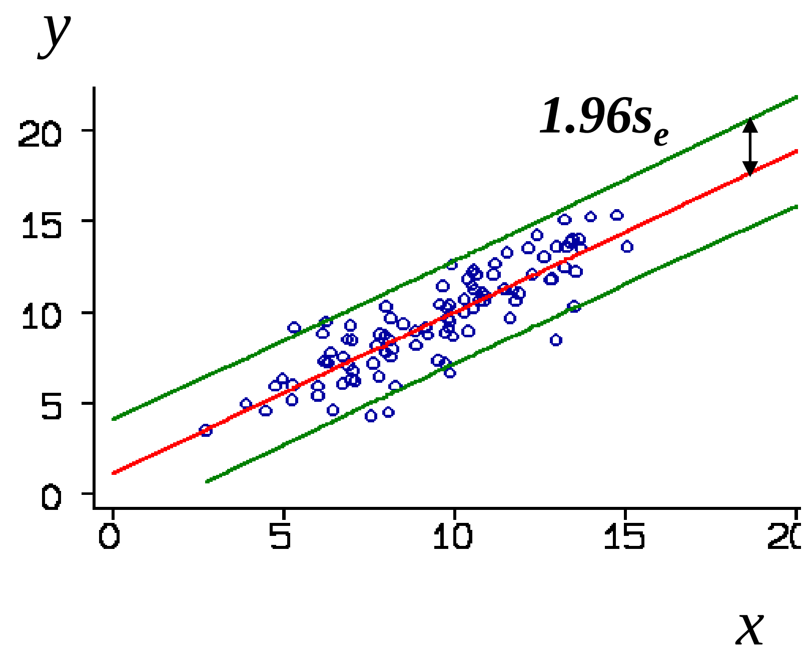
Correct use of regression

- Show the 95% confidence interval
  - Calculate the standard deviation of the difference s_e
  - trace the zones of $\pm 1.96\, s_e$ (normal distribution, z = 1.96 for α = 2.5%)

$$s_e = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - y_{ei})^2}$$

point in the data set

point on the line



Hypothesis test

Q1: Does the average value supplied by the sensor correspond to the actual value we are trying to measure?

How do we answer this?

By doing a hypothesis test which consists of:

- Making the initial assumption
- Collecting evidence (data)
- Based on the available evidence (data), deciding whether to reject or not reject the initial assumption.

Hypothesis test

Q1: Does the average value supplied by the sensor correspond to the actual value?

- We know the theoretical average (actual value, μ) of a population and its confidence interval:

$$\mu - z_\alpha \frac{\sigma}{\sqrt{N}} < \mu < \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$$

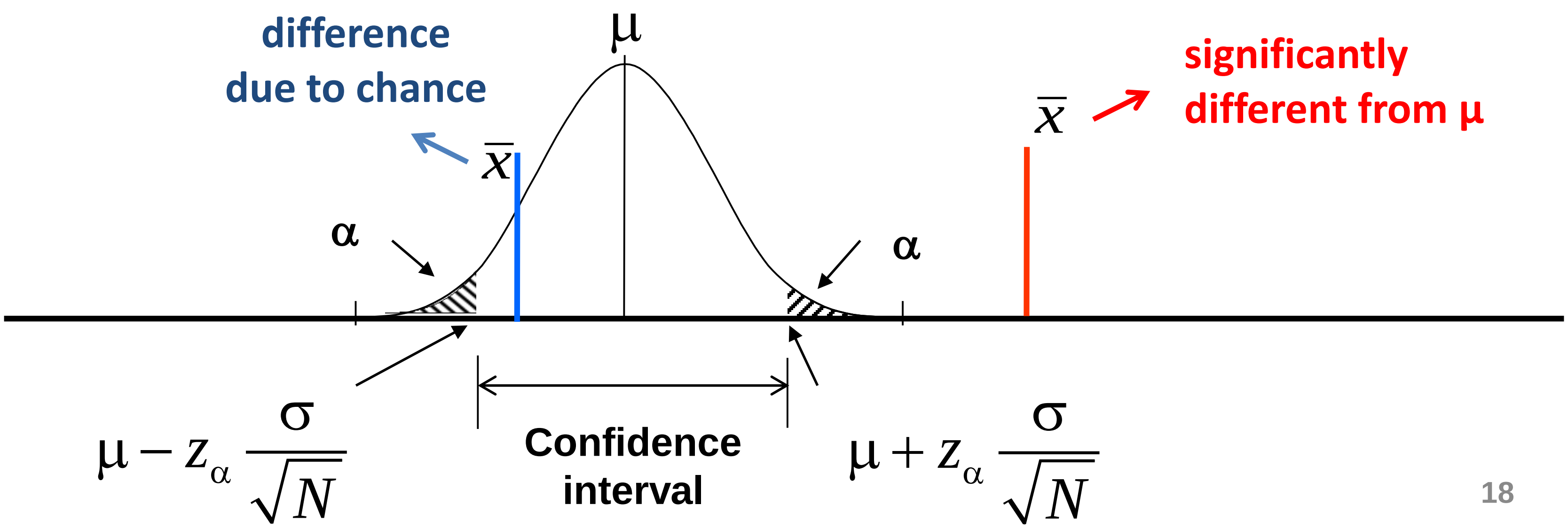
- We collect N samples from this population (make N measurements), calculate the average and find:

$$\bar{x} \neq \mu$$

- The question now is whether the average $\bar{x}$ is significantly different from the actual value μ or if the difference is due to chance

Hypothesis test

- $\bar{x}$ is significantly different from the actual value μ -
  - $\bar{x}$ is outside the confidence interval
- The difference between $\bar{x}$ and μ is due to chance
  - $\bar{x}$ is inside the confidence interval



Hypothesis test - example

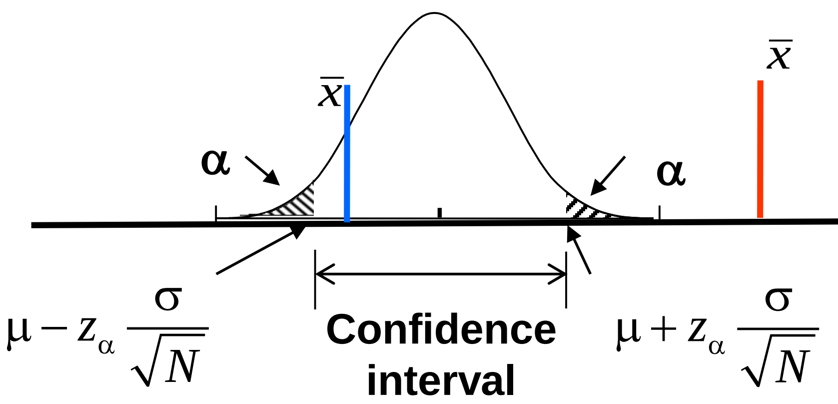
- Specification sheet: the voltage source provides μ = 10 V with a σ = 0.2 V (This is the claim, hypothesis)
- How do we test this?
  - Perform N = 100 measurements (N can be any large number of measurements > 30 so we can apply the normal distribution) and calculate $\bar{x}$
  - Estimate if $\bar{x}$ is in the specified confidence interval

- If $\bar{x} > \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$ or $\bar{x} < \mu - z_\alpha \frac{\sigma}{\sqrt{N}}$

  - systematic error

- If $\mu - z_\alpha \frac{\sigma}{\sqrt{N}} \leq \bar{x} \leq \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$

  - difference due to chance



Hypothesis test - example

- For p = 95% $z_{\alpha=0.025}$=1.96
- For $\bar{x}$=10.1V

$$z_{obs} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}} = \frac{10.1 - 10}{0.2/10} = 5 > z_\alpha$$

- We **reject** the hypothesis (claim) of the manufacturer. We are 95% sure that the difference between $\bar{x}$=10.1V and μ=10V is significant.

- For $\bar{x}$=10.03V

$$z'_{obs} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}} = \frac{10.03 - 10}{0.2/10} = 1.5 < z_\alpha$$

- We **do not reject** the hypothesis of the manufacturer. We can however not claim that the source actually delivers 10V, only that the difference is due to chance (random error).
- A hypothesis is <u>never accepted</u>: only **rejected** or **not rejected**.

Usefulness of the hypothesis test

- How do we know if the hypothesis on the measured values is right or probable
- Example: Hypothesis – the value provided by the voltage source is 10.01 V
- How do we check this?
  - do an infinitely large number of measurements and calculate the average
  - perform sampling and calculate the average – SIGNIFICANCE
- Significant difference: the difference between two values is not due to chance **(systematic error)**
- No significant difference: the difference is due to chance **(random error)**

Main uses of hypothesis testing

- Comparison of an experimental average with a theoretical one
- Comparison of two experimental averages
- Comparison of two variances (precisions)
- Comparison of an experimental variance with a theoretical variance

Definition of the hypothesis test

- Data analysis procedure with the outcome of **rejecting** or **failing to reject** (not the same as accepting!) a hypothesis based on the data
- There are always two hypotheses:
  - **H_0** – the result of an estimation <u>does not significantly differ</u> from the actual value (theoretical or supposed). This is the **null hypothesis**

    H_0: $\bar{x} = \mu$ there is no difference between the estimated average and the theoretical value

  - **H_a** – the result of an estimation <u>significantly differs</u> from the actual value (theoretical or supposed). **This is the alternative hypothesis.**

    H_a: $\bar{x} \neq \mu$

- During a hypothesis test, we always assume **H_0 is true** and announce it in the form of a sentence: Example – Can we say that the voltage source provides 10V?

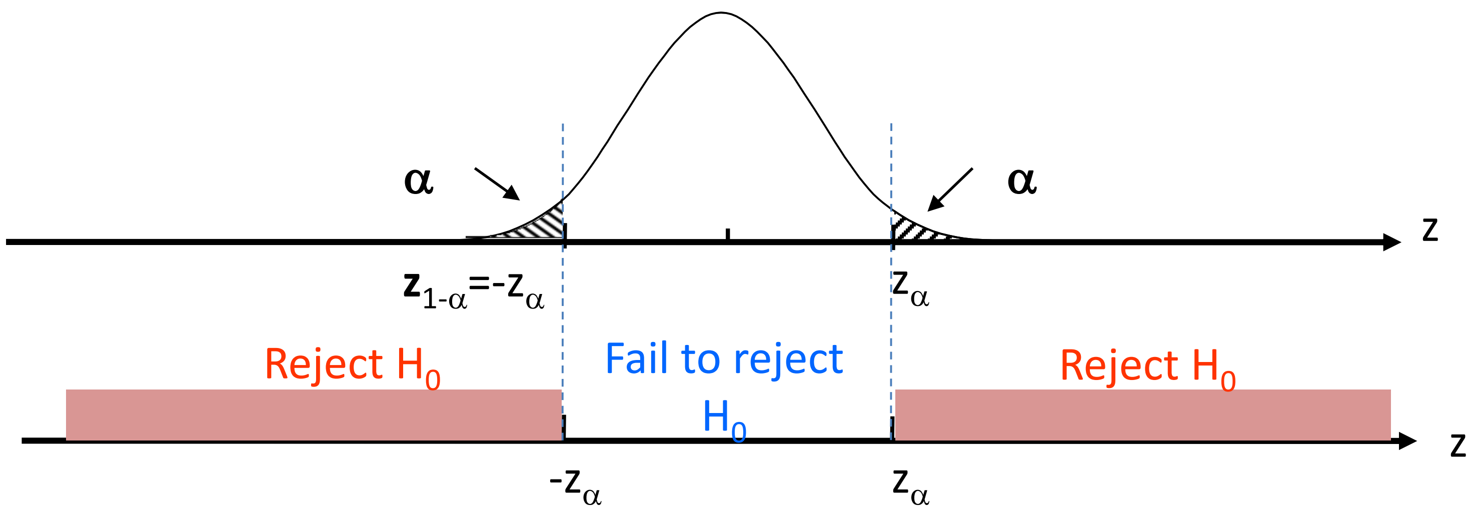
  H_0 – **"**The estimated average is not different from 10V"

  H_0 – "The difference between the estimated value and 10V is zero."

  H_0 – "The voltage source provides 10V."

Bilateral test

- Used when we do not know in advance the particular direction of the alternative hypothesis (if $\bar{x} > \mu$ or $\bar{x} < \mu$)
- Use α to determine the risk of error which is β=2α



Flowchart for the realisation of a bilateral test

Formulate the hypothesis H_0

A sentence expressing the lack of difference:
"The voltage source is providing 10V"
or "The tension of the voltage source is not different from 10V"

↓

Define the statistical distribution (z)

↓

Set the risk $\beta=2\alpha$

in general 1%, 5%, 10%

↓

Calculate z_α

tables

↓

Calculate z_{obs}

$$z_{obs} = \frac{\bar{x}-\mu}{\sigma/\sqrt{N}}$$

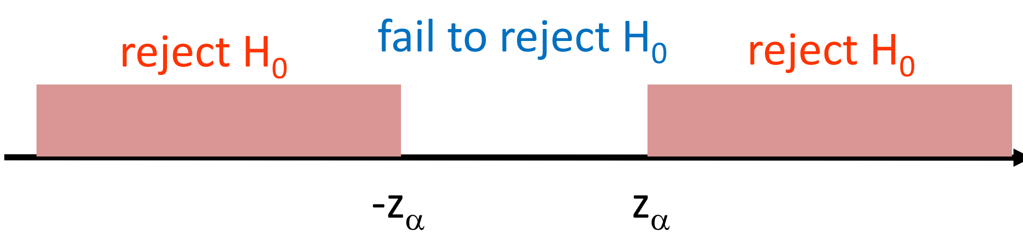
↓

Compare with $CI_{(1-2\alpha)\%}$

If $|z_{obs}| > |z_\alpha|$ **– reject H_0**

↓

Formulate the conclusion



Example: bilateral z-test

- According to the specifications, a sensor should draw 2.80 mA of current with a standard deviation of 0.14 mA. To test this, we take 40 sensors and find an average current draw of 2.72 mA. What can we conclude with a risk of 5% about the specifications?

H_0: μ=2.80 mA – The sensor draws 2.80 mA

$$z_{obs} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}}$$

α = 5%/2 = 0.025

z_α = - 1.96

z_{obs}= (2.72-2.80)/(0.14/6.3)=-3.61

-3.61<-1.96 : we reject Ho

Conclusion: the average current draw is different from 2.8 mA, with a risk of 5%.

,8531	−4,265	,00001
,8643	−3,719	,0001
,8749	−3,090	,001
,8849	−2,576	,005
,8944	−2,326	,01
,9032	−2,054	,02
,9115	$\boxed{-1,960}$	$\boxed{,025}$
,9192	−1,881	,03
,9265	−1,751	,04
9332	−1,645	,05
,9394	−1,555	,06
,9452	−1,476	,07
,9505	−1,405	,08
,9554	−1,341	,09
,9599	−1,282	,10

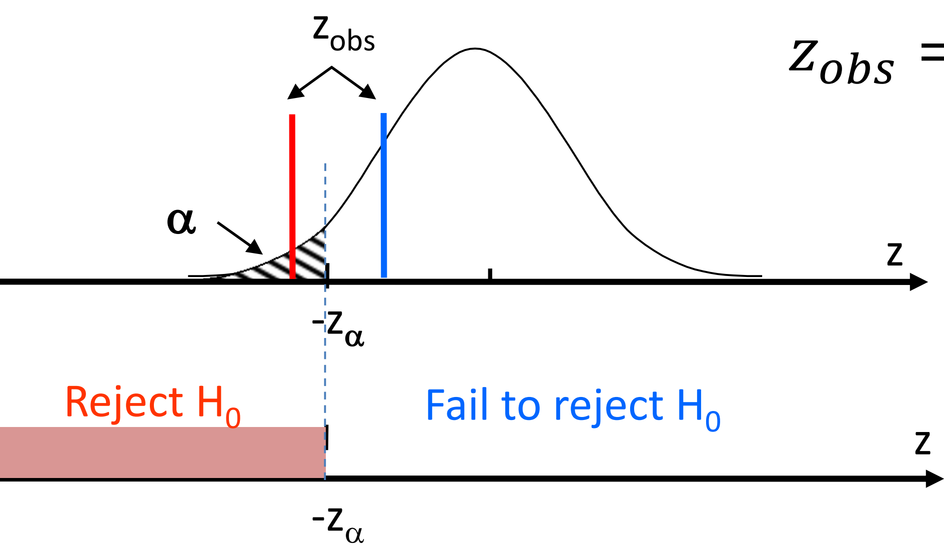
Unilateral test

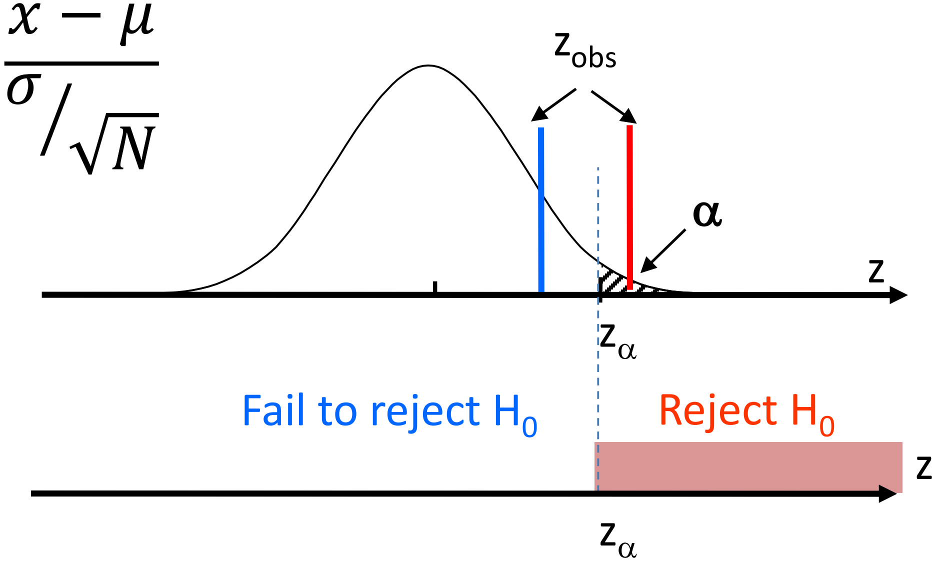
- Used when we expect the average to be above or below the theoretical average (specifications)
- Use α to determine the risk of error (in this case it's **α!**)
- Formulate the null hypothesis

H$_0$: $\bar{x} > \mu$ therefore H$_a$: $\bar{x} < \mu$

H$_0$: $\bar{x} < \mu$ therefore H$_a$: $\bar{x} > \mu$

$$z_{obs} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}}$$





Flowchart for the realisation of a unilateral test

Formulate the hypothesis H_0:

$$\bar{x} \geq \mu$$

Define the statistical distribution (z)

Set the risk α

Calculate z_α

Calculate $z_{obs} = \frac{\bar{x}-\mu}{\sigma/\sqrt{N}}$

Compare with $CI_{(1-\alpha)\%}$

If $z_{obs} < -|z_\alpha|$ **– reject H_0**

Formulate the conclusion

Formulate the hypothesis H_0:

$$\bar{x} \leq \mu$$

Define the statistical distribution (z)

Set the risk α

Calculate z_α

Calculate $z_{obs} = \frac{\bar{x}-\mu}{\sigma/\sqrt{N}}$

Compare with $CI_{(1-\alpha)\%}$

If $z_{obs} > |z_\alpha|$ **– reject H_0**

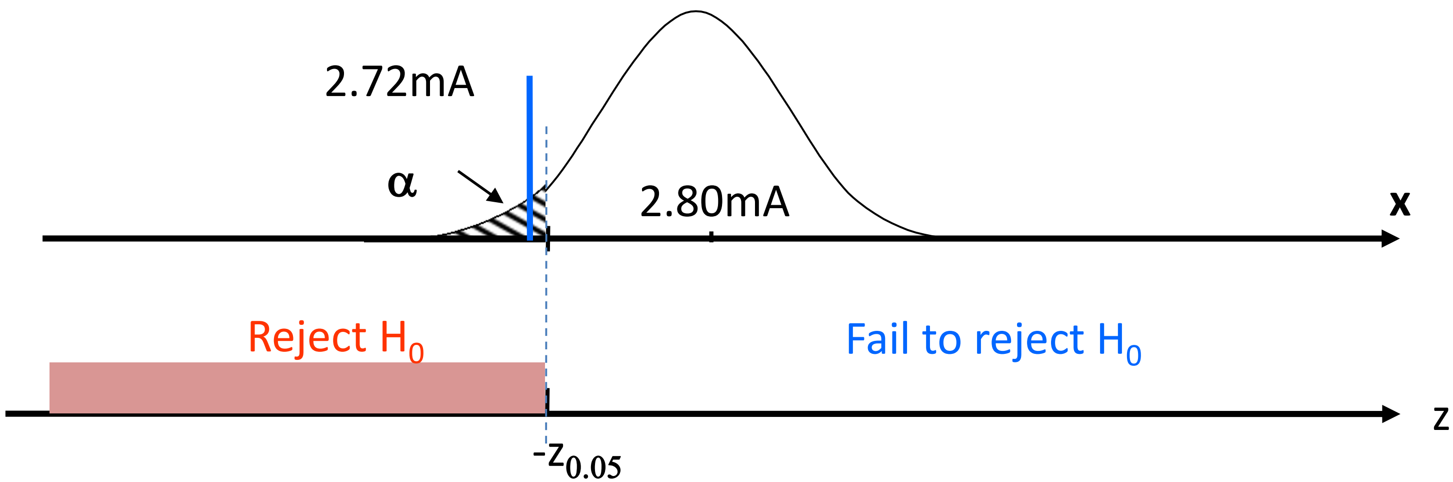
Formulate the conclusion

Example: unilateral z-test

- According to the specifications, a sensor should draw 2.80 mA of current with a standard deviation of 0.14 mA. To test this, we take 40 sensors and find an average current draw of 2.72 mA. What can we conclude about the specifications with a 5% risk?
- Express the null hypothesis

  H_0 : the current draw is higher than 2.80 mA

  (because the only thing we can do is reject a hypothesis, we can't accept it)



Example: unilateral z-test

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,9115	−1,960	,025
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,9265	−1,751	,04
9332	$\boxed{-1,645 \quad ,05}$	
,9394	−1,555	,06
,9452	−1,476	,07
,9505	−1,405	,08
,9554	−1,341	,09
,9599	−1,282	,10

H_0: $\boldsymbol{\overline{x} \geq \mu}$ the current draw is higher than 2.80 mA

$$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{N}}$$

$\alpha = 5\% = 0.05$

$z_\alpha = -1.645$

$z_{obs} = (2.72-2.80)/(0.14/6.3) = -3.61$

$-3.61 < -1.645$: we reject Ho

Conclusion: the average current draw is smaller than 2.8 mA, with a risk of 5%.

Comparison of an experimental average with a theoretical one – t-test

- In cases where the number of measurements N < 30 or if the standard deviation is estimated from an experiment (and not specifications), we use the Student distribution instead of the normal one and the t-test instead of the z-test

- The procedure is the same as in the z-test:
  - we replace z_α with t_α, defined by the Student distribution
  - t_{obs} is given by:

$$t_{obs} = \frac{\bar{x} - \mu}{s/\sqrt{N}}$$

Example: t-test

- In order to estimate the error of a gyroscope reading, we carry out 22 identical tests. They consist of turning the gyroscope by 360°, taking the angular velocity readings while the gyroscope is turning and then integrating the velocity (which should give us the total angle or 360°). We find in this way an average value of 359.2° and a standard deviation of 4.4°. Can we say with a 5% risk that the sensor is producing a systematic error?

H_0: The sensor is not making a systematic error
The difference between the result and the theoretical value is not significant

$$t = \frac{\bar{x} - \mu}{s/\sqrt{N}}$$

$2\alpha = 5\%$; $\alpha = 0.025$ (bilateral test)
$t_\alpha = 2.08$
$t_{obs} = (359.2-360)/(4.4/4.7) = -0.85$
$-0.85 > -2.08$: we do not reject H_0

17	,257	,689	1,333	1,740	2,110
18	,257	,688	1,330	1,734	2,101
19	,257	,688	1,328	1,729	2,093
20	0,257	0,687	1,325	1,725	2,086
21	,257	,686	1,323	1,721	$\boxed{2,080}$
22	,256	,686	1,321	1,717	2,074
23	,256	,685	1,319	1,714	2,069
24	,256	,685	1,318	1,711	2,064

Conclusion: the difference is not significant with a risk of 5%.

Comparison of two experimental averages: z-test

- Comparing $\overline{x_1}$ with $\overline{x_2}$ is the equivalent of comparing $\overline{x_1} - \overline{x_2}$ with 0.
- If x_1 and x_2 are independent, then:

$$Var(\bar{x}_1 - \bar{x}_2) = Var(\bar{x}_1) + Var(\bar{x}_2) = \frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}$$

- The variable z follows the normal distribution $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2/N_1 + \sigma_2^2/N_2}}$
- If the standard deviations have been determined from the experiment or $N_1 < 30$ or $N_2 < 30$, we use the variable t and the Student distribution

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/N_1 + s_2^2/N_2}}$$

Example

- Two sensors have noisy outputs with U_1 = 9.8 V (σ_1 = 1V) and U_2 = 9.6 V (σ_2 = 1.32 V). Can we say with a risk of 2% that the values U_1 and U_2 are different after taking 500 measurements?

H_0: The two averages are not different

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2/N_1 + \sigma_2^2/N_2}}$$

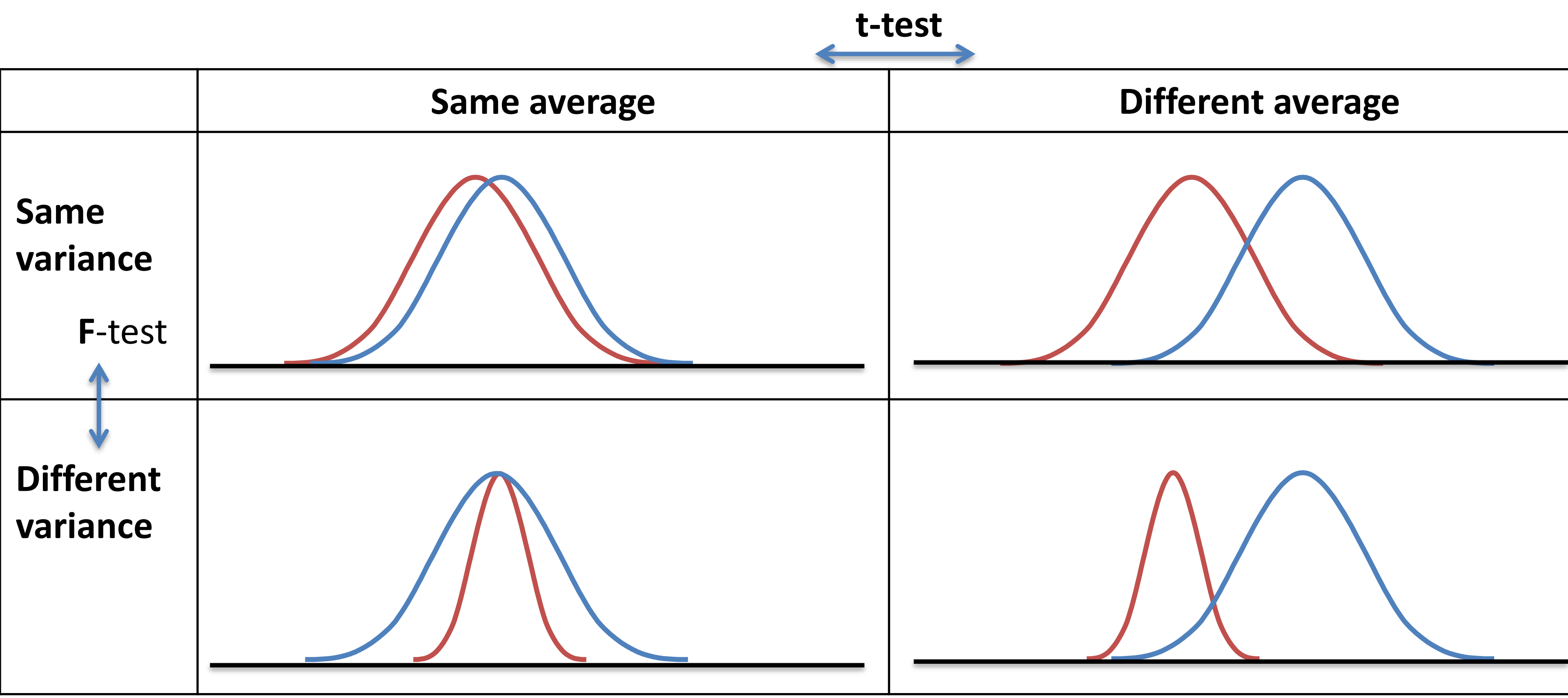
z	α	z	α
1,05	,8531	−4,265	,00001
1,10	,8643	−3,719	,0001
1,15	,8749	−3,090	,001
1,20	,8849	−2,576	,005
1,25	,8944	$\boxed{-2,326}$	,01
1,30	,9032	−2,054	,02
1,35	,9115	−1,960	,025
1,40	,9192	−1,881	,03
1,45	,9265	−1,751	,04

α = 2%/2 = 0.01 (bilateral test)
$z_{1\%}$ = - 2.33
z_{obs}= (9.8-9.6)/(1.66/22.4)=2.69
2.69>2.33 : we reject H_0

Conclusion: the averages are significantly different, with a risk of 2%.

Comparison between two variances

- We can also compare two sets of data according to their variance

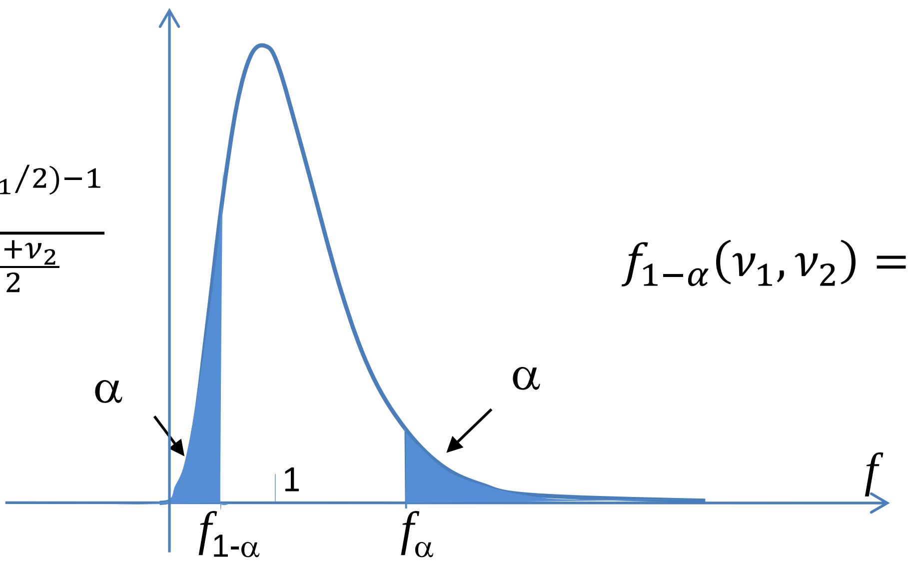


Comparing two measured variances

- Let us assume that we have two sets of data (populations) with distributions that can be described using the normal (Gaussian) distribution
- Let s_1^2 and s_2^2 be the variances estimated using N_1 and N_2 samples with $s_1^2 > s_2^2$ (s not σ, to stress that the variance is measured)
- In this case the quantity $f = \frac{s_1^2}{s_2^2}$ follows the Fisher distribution

$f(\nu_1, \nu_2)$

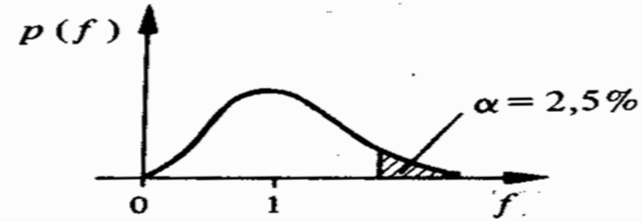
$$p(f) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right) {\nu_1}^{\nu_1/2} {\nu_2}^{\nu_2/2} f^{(\nu_1/2)-1}}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)(\nu_2+\nu_1 f)^{\frac{\nu_1+\nu_2}{2}}}$$



$$\nu_1 = N_1 - 1$$

$$\nu_2 = N_2 - 1$$

$$f_{1-\alpha}(\nu_1, \nu_2) = 1/f_\alpha(\nu_2, \nu_1)$$



ν_2 \ ν_1	1	2	12	15	20	24	30	40	60	120
1	647,8	799,5	976,7	984,9	993,1	997,2	1001	1006	1010	1014
2	38,51	39,00	39,41	39,43	39,45	39,46	39,46	39,47	39,48	39,49
3	17,44	16,04	14,34	14,25	14,17	14,12	14,08	14,04	13,99	13,95
4	12,22	10,65	8,75	8,66	8,56	8,51	8,46	8,41	8,36	8,31
5	10,01	8,43	6,52	6,43	6,33	6,28	6,23	6,18	6,12	6,07
6	8,81	7,26	5,37	5,27	5,17	5,12	5,07	5,01	4,96	4,90
7	8,07	6,54	4,67	4,57	4,47	4,42	4,36	4,31	4,25	4,20
8	7,57	6,06	4,20	4,10	4,00	3,95	3,89	3,84	3,78	3,73
9	7,21	5,71	3,87	3,77	3,67	3,61	3,56	3,51	3,45	3,39
10	6,94	5,46	3,62	3,52	3,42	3,37	3,31	3,26	3,20	3,14
11	6,72	5,26	3,43	3,33	3,23	3,17	3,12	3,06	3,00	2,94
12	6,55	5,10	3,28	3,18	3,07	3,02	2,96	2,91	2,85	2,79
13	6,41	4,97	3,15	3,05	2,95	2,89	2,84	2,78	2,72	2,66
14	6,30	4,86	3,05	2,95	2,84	2,79	2,73	2,67	2,61	2,55
15	6,20	4,77	2,96	2,86	2,76	2,70	2,64	2,59	2,52	2,46
16	6,12	4,69	2,89	2,79	2,68	2,63	2,57	2,51	2,45	2,38
17	6,04	4,62	2,82	2,72	2,62	2,56	2,50	2,44	2,38	2,32
18	5,98	4,56	2,77	2,67	2,56	2,50	2,44	2,38	2,32	2,26
19	5,92	4,51	2,72	2,62	2,51	2,45	2,39	2,33	2,27	2,20
20	5,87	4,46	2,68	2,57	2,46	2,41	2,35	2,29	2,22	2,16
21	5,83	4,42	2,64	2,53	2,42	2,37	2,31	2,25	2,18	2,11
22	5,79	4,38	2,60	2,50	2,39	2,33	2,27	2,21	2,14	2,08
23	5,75	4,35	2,57	2,47	2,36	2,30	2,24	2,18	2,11	2,04
24	5,72	4,32	2,54	2,44	2,33	2,27	2,21	2,15	2,08	2,01
25	5,69	4,29	2,51	2,41	2,30	2,24	2,18	2,12	2,05	1,98
26	5,66	4,27	2,49	2,39	2,28	2,22	2,16	2,09	2,03	1,95
27	5,63	4,24	2,47	2,36	2,25	2,19	2,13	2,07	2,00	1,93
28	5,61	4,22	2,45	2,34	2,23	2,17	2,11	2,05	1,98	1,91
29	5,59	4,20	2,43	2,32	2,21	2,15	2,09	2,03	1,96	1,89
30	5,57	4,18	2,41	2,31	2,20	2,14	2,07	2,01	1,94	1,87
40	5,42	4,05	2,29	2,18	2,07	2,01	1,94	1,88	1,80	1,72
60	5,29	3,93	2,17	2,06	1,94	1,88	1,82	1,74	1,67	1,58
120	5,15	3,80	2,05	1,94	1,82	1,76	1,69	1,61	1,53	1,43
∞	5,02	3,69	1,94	1,83	1,71	1,64	1,57	1,48	1,39	1,27

Example: bilateral F-test

- Two sensors have noisy outputs with $U_1 = 9.35$ V ($s_1 = 1.5$V) and $U_2 = 9.8$ V ($s_2 = 1$ V). Can we say with a risk of 5% that the two sensors have different noise levels? s_1 and s_2 have been calculated based on 31 measurements.

H_0: The two noise levels are not different

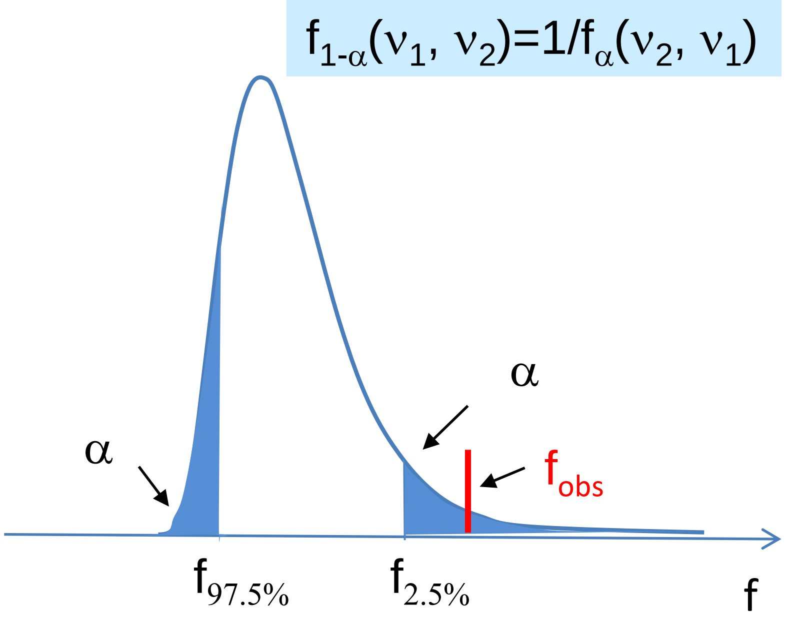
$$f = \frac{s_1^2}{s_2^2}$$

$\alpha = 5\%/2 = 0.025$ (bilateral test)

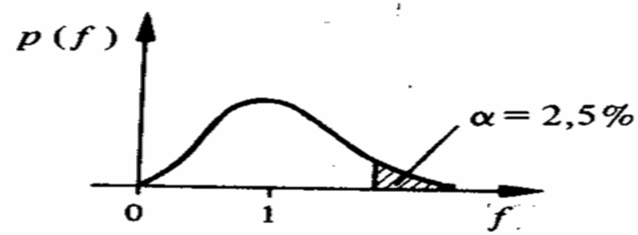
$f_{1-\alpha}(\nu_1, \nu_2)=1/f_{\alpha}(\nu_2, \nu_1)$

$f_{2.5\%}(30,30)=2.07$, $f_{97.5\%}(30,30)=0.483$

$f_{obs}=(1.5)^2/1=2.25 > 2.07$: we reject H_0



Conclusion: the noise levels are significantly different, with a risk of 5%.



ν_1	ν_2 1	2	12	15	20	24	30	40	60	120
1	647,8	799,5	976,7	984,9	993,1	997,2	1001	1006	1010	1014
2	38,51	39,00	39,41	39,43	39,45	39,46	39,46	39,47	39,48	39,49
3	17,44	16,04	14,34	14,25	14,17	14,12	14,08	14,04	13,99	13,95
4	12,22	10,65	8,75	8,66	8,56	8,51	8,46	8,41	8,36	8,31
5	10,01	8,43	6,52	6,43	6,33	6,28	6,23	6,18	6,12	6,07
6	8,81	7,26	5,37	5,27	5,17	5,12	5,07	5,01	4,96	4,90
7	8,07	6,54	4,67	4,57	4,47	4,42	4,36	4,31	4,25	4,20
8	7,57	6,06	4,20	4,10	4,00	3,95	3,89	3,84	3,78	3,73
9	7,21	5,71	3,87	3,77	3,67	3,61	3,56	3,51	3,45	3,39
10	6,94	5,46	3,62	3,52	3,42	3,37	3,31	3,26	3,20	3,14
11	6,72	5,26	3,43	3,33	3,23	3,17	3,12	3,06	3,00	2,94
12	6,55	5,10	3,28	3,18	3,07	3,02	2,96	2,91	2,85	2,79
13	6,41	4,97	3,15	3,05	2,95	2,89	2,84	2,78	2,72	2,66
14	6,30	4,86	3,05	2,95	2,84	2,79	2,73	2,67	2,61	2,55
15	6,20	4,77	2,96	2,86	2,76	2,70	2,64	2,59	2,52	2,46
16	6,12	4,69	2,89	2,79	2,68	2,63	2,57	2,51	2,45	2,38
17	6,04	4,62	2,82	2,72	2,62	2,56	2,50	2,44	2,38	2,32
18	5,98	4,56	2,77	2,67	2,56	2,50	2,44	2,38	2,32	2,26
19	5,92	4,51	2,72	2,62	2,51	2,45	2,39	2,33	2,27	2,20
20	5,87	4,46	2,68	2,57	2,46	2,41	2,35	2,29	2,22	2,16
21	5,83	4,42	2,64	2,53	2,42	2,37	2,31	2,25	2,18	2,11
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24	5,72	4,32	2,54	2,44	2,33	2,27	2,21	2,15	2,08	2,01
25	5,69	4,29	2,51	2,41	2,30	2,24	2,18	2,12	2,05	1,98
26	5,66	4,27	2,49	2,39	2,28	2,22	2,16	2,09	2,03	1,95
27	5,63	4,24	2,47	2,36	2,25	2,19	2,13	2,07	2,00	1,93
28	5,61	4,22	2,45	2,34	2,23	2,17	2,11	2,05	1,98	1,91
29	5,59	4,20	2,43	2,32	2,21	2,15	2,09	2,03	1,96	1,89
30	5,57	4,18	2,41	2,31	2,20	2,14	$\boxed{2,07}$	2,01	1,94	1,87
40	5,42	4,05	2,29	2,18	2,07	2,01	1,94	1,88	1,80	1,72
60	5,29	3,93	2,17	2,06	1,94	1,88	1,82	1,74	1,67	1,58
120	5,15	3,80	2,05	1,94	1,82	1,76	1,69	1,61	1,53	1,43
∞	5,02	3,69	1,94	1,83	1,71	1,64	1,57	1,48	[illegible]	[illegible]

Comparing two measured variances – F-test

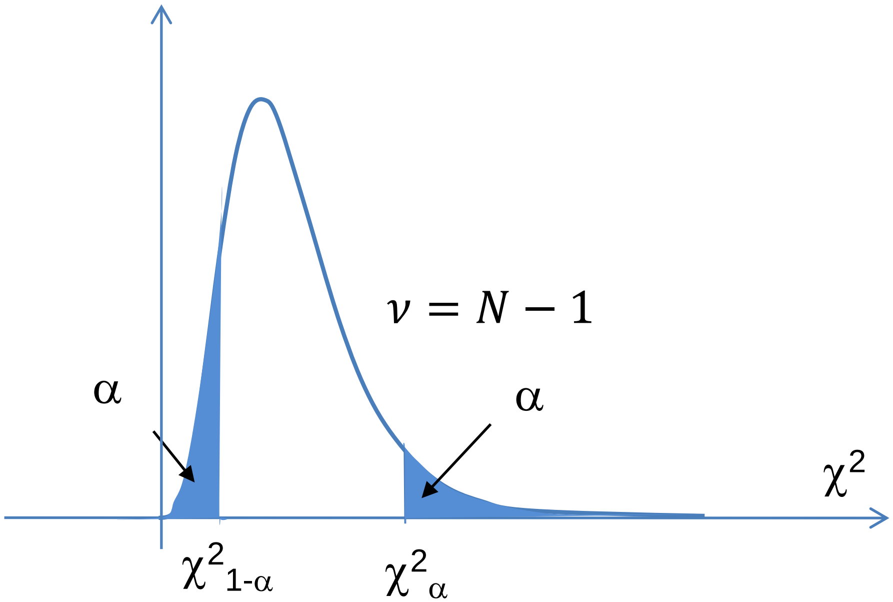
bilateral	unilateral
Formulate the hypothesis H_0 $s_1 = s_2$	**Formulate the hypothesis H_0 $s_1 \geq s_2$**
Define the test variable (*f*)	Define the test variable (*f*)
Set the risk **2α**	Set the risk **α**
Calculate f_α and $f_{1-\alpha}$	Calculate $f_{1-\alpha}$
Calculate $f_{obs} = \frac{s_1^2}{s_2^2}$	Calculate $f_{obs} = \frac{s_1^2}{s_2^2}$
Compare with $CI_{(1-2\alpha)\%}$ ***If*** $f_{obs} > f_\alpha$ **or** $f_{obs} < f_{1-\alpha}$ **– reject H_0**	Compare with $CI_{(1-\alpha)\%}$ ***If*** $f_{obs} < f_{1-\alpha}$ **– reject H_0**
Formulate the conclusion	Formulate the conclusion

Comparison between an experimental and a theoretical variance: χ^2 test

- Let σ^2 be the theoretical variance and s^2 the experimentally determined variance, estimated using N samples

- The variable χ^2 is defined as $\chi^2 = \frac{(N-1)s^2}{\sigma^2}$ with a distribution $\chi^2(\nu)$

$$p(\chi^2) = \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right) \right]^{-1} (\chi^2)^{(\nu/2)-1} e^{\left(-\frac{\chi^2}{2}\right)}$$



Example: χ^2 test

- An amplifier is characterised by noise $\sigma = 2.2$ µV. A filter is used at the output in order to reduce this noise. The noise amplitude after filtering is estimated to be $s = 1.92$ µV based on 31 measurements. Determine with a risk of 5% if the filter is effective in reducing the noise.

H_0: The noise level after filtering is higher than before the filtering

$$\chi^2 = \frac{(N-1)s^2}{\sigma^2}$$

α = 5% (unilateral test)

$\chi^2_{95\%}(30)=18.49$

$\chi^2_{obs}=(30)\times 1.92^2/2.2^2 = 22.85$

$22.85 > 18.49$: we do not reject H_0

Conclusion: the filter is not effective

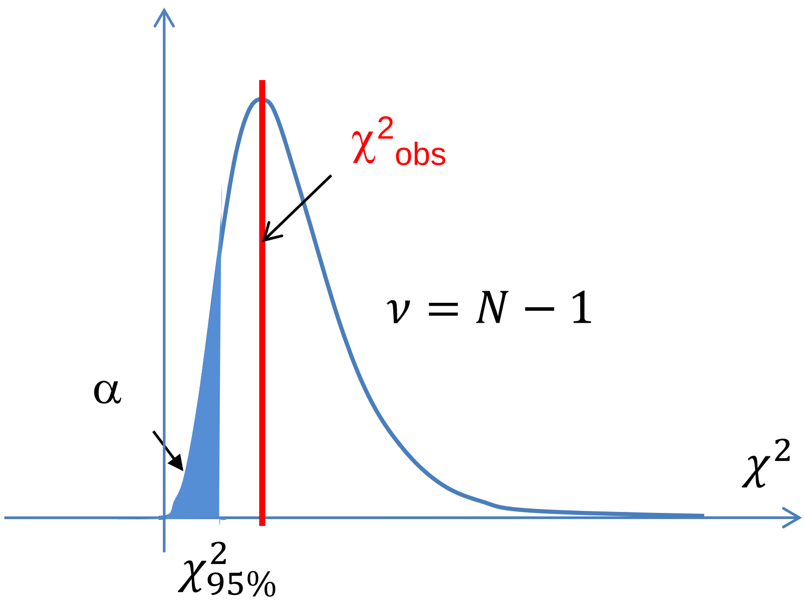


Table 5 Values of χ^2_α *

ν	$\alpha = 0.995$	$\alpha = 0.99$	$\alpha = 0.975$	$\alpha = 0.95$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	ν
1	0.0000393	0.000157	0.000982	0.00393	3.841	5.024	6.635	7.879	1
2	0.0100	0.0201	0.0506	0.103	5.991	7.378	9.210	10.597	2
3	0.0717	0.115	0.216	0.352	7.815	9.348	11.345	12.838	3
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860	4
5	0.412	0.554	0.831	1.145	11.070	12.832	15.056	16.750	5
6	0.676	0.872	1.237	1.635	12.592	14.449	16.812	18.548	6
7	0.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	10
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	11
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	12
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819	13
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	14
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	15
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267	16
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	17
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156	18
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	19
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	20
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401	21
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796	22
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181	23
24	9.886	10.856	12.401	13.484	36.415	39.364	42.980	45.558	24
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928	25
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290	26
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645	27
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993	28
29	13.121	14.256	16.047	17.708	42.557	45.772	49.588	52.336	29
30	13.787	14.953	16.791	$\boxed{18.493}$	43.773	46.979	50.892	53.672	30
40	20.706	22.164	24.433	26.509	55.758	59.342	63.691	66.766	40
50	27.991	29.707	32.357	34.764	67.505	71.420	76.154	79.490	50
60	35.535	37.485	40.482	43.118	79.082	83.298	88.379	91.952	60
70	43.275	45.442	48.758	51.739	90.531	95.023	100.425	104.215	70
80	51.172	53.540	57.153	60.391	101.879	106.629	112.329	116.321	80
90	59.196	61.754	65.646	69.126	113.145	118.136	124.116	128.299	90
100	67.328	70.065	74.222	77.929	124.342	129.561	135.807	140.169	100

Comparison between an experimental and a theoretical variance: bilateral χ^2 test

Formulate the hypothesis H_0
$\boldsymbol{s = \sigma}$

Define the test variable (χ^2)

Set the risk **2α**

Calculate χ^2_{α} and $\chi^2_{1-\alpha}$

Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$

Compare with $CI_{(1-2\alpha)\%}$

If $\chi^2_{obs} > \chi^2_{\alpha}$ **or** $\chi^2_{obs} < \chi^2_{1-\alpha}$
reject H_0

Formulate the conclusion

Comparison between an experimental and a theoretical variance: unilateral χ^2test

Formulate the hypothesis H_0 $\boldsymbol{s \geq \sigma}$

Define the test variable (χ^2)

Set the risk **α**

Calculate $\chi^2_{1-\alpha}$

Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$

Compare with $CI_{(1-\alpha)\%}$

If $\chi^2_{obs} < \chi^2_{1-\alpha}$ **reject H_0**

Formulate the conclusion

Formulate the hypothesis H_0 $\boldsymbol{s \leq \sigma}$

Define the test variable (χ^2)

Set the risk **α**

Calculate χ^2_{α}

Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$

Compare with $CI_{(1-\alpha)\%}$

If $\chi^2_{obs} > \chi^2_{\alpha}$ **reject H_0**

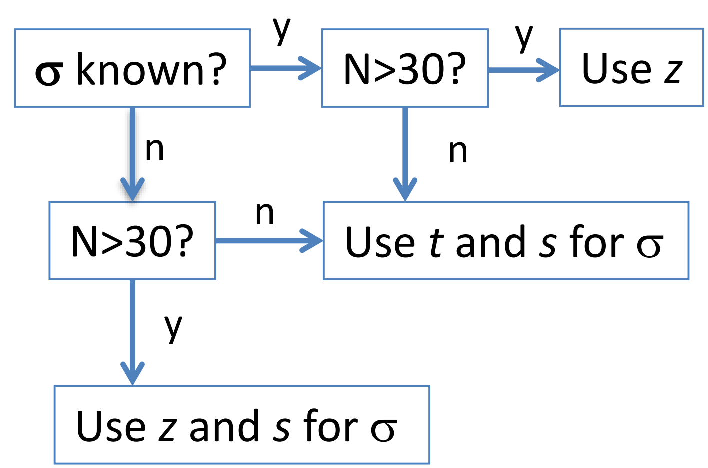
Formulate the conclusion

General procedure for a bilateral test

Formulate the hypothesis H_0 — for example $\bar{x} = \mu$ or $s_1 = s_2$, etc.

Define the test statistics (q) — for example $q = z$, $q = t$, etc

y = yes
n = no



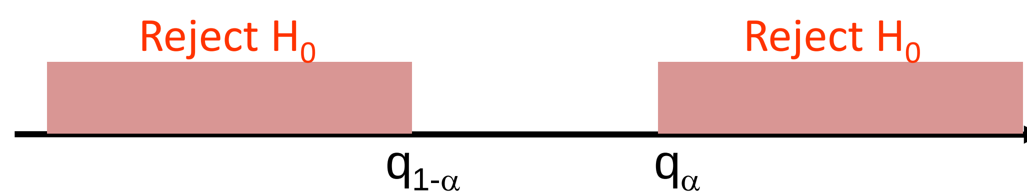
Set the risk **2α**

Calculate q_α and $q_{1-\alpha}$ — tables

Calculate q_{obs} — $$z_{obs} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}}$$

Compare with $CI_{(1-2\alpha)\%}$

If $q_{obs} > q_\alpha$ or $q_{obs} < q_{1-\alpha}$ – reject H_0



Formulate the conclusion

General procedure for a unilateral test

Formulate the hypothesis H$_0$ $\bar{x} \geq \mu$

Define the test variable (q)

Set the risk **α**

Calculate $q_{1-\alpha}$

Calculate q_{obs}

Compare with CI$_{(1\text{-}\alpha)\%}$

If $q_{obs} < q_{1-\alpha}$ **reject H$_0$**

Formulate the conclusion

Formulate the hypothesis H$_0$ $\bar{x} \leq \mu$

Define the test variable (q)

Set the risk **α**

Calculate q_{α}

Calculate q_{obs}

Compare with CI$_{(1\text{-}\alpha)\%}$

If $q_{obs} > q_{\alpha}$ **reject H$_0$**

Formulate the conclusion

Key points

	Comparison	Test	Conditions	Variable
AVERAGES	Theoretical vs. experimental	z-test	$\sigma_{theoretical}$ known and N>30	$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}}$
		t-test	$s_{experimental}$ known or N<30	$t = \frac{\bar{x} - \mu}{s / \sqrt{N}}$
	Experimental vs. experimental	z-test	z follows a normal distribution $N_1 \geq 30$ and $N_2 \geq 30$	$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{{\sigma_1}^2 / N_1 + {\sigma_2}^2 / N_2}}$
		t-test	$N_1 < 30$ or $N_2 < 30$	$t = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{{s_1}^2 / N_1 + {s_2}^2 / N_2}}$
VARIANCES	Theoretical vs. experimental	χ2-test		$\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$
	Experimental vs. experimental	F-test		$f_{obs} = \frac{s_1^2}{s_2^2}$