Exercise 1 – Beam Hardening

The polychromatic x-rays used in x-ray imaging and CT experience different attenuations for different wave energies; this is due to the fact that the attenuation in biological tissue is dependent on the energy of the rays. This leads to the effect called "beam hardening".

a) A typical energy distribution of the beam from the x-ray source is shown in Figure 1. Sketch the energy spectrum after the beam has passed through the body.

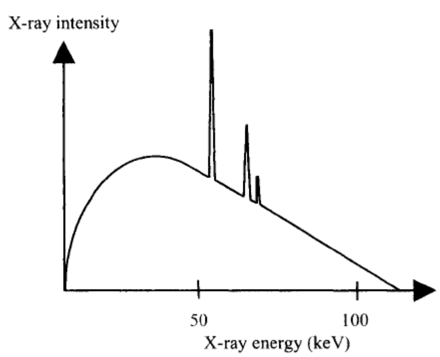


Figure 1

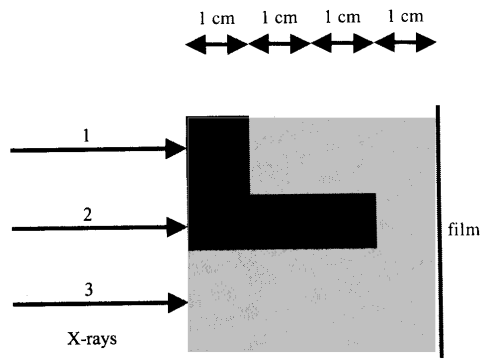


Figure 2

b) In Figure 2, calculate the X-ray intensity, as a function of the incident intensity I_0 that reaches the film for each of the three X-ray beams. The dark-shaded area represents bone and the light-shaded area represents tissue. The linear attenuation coefficients at the effective X-ray energy of 68keV are 10 and 1 cm^{-1} for bone and tissue, respectively.

Exercise 2 – Sinogram

A straightforward depiction of the data obtained by a CT scanner is a sinogram. It shows the imaging data accumulated by the different projections.

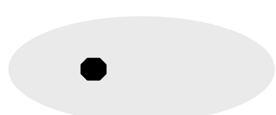


Figure 3

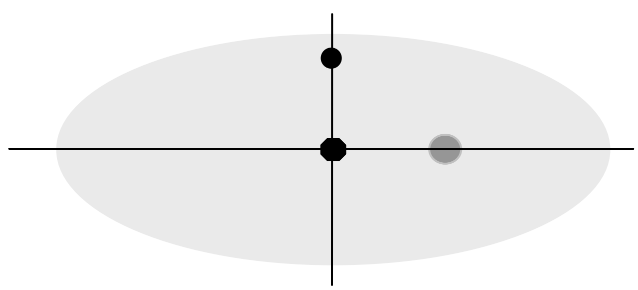


Figure 4

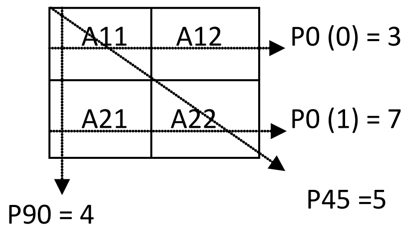
a) For the object shown in Figure 3, draw the CT projections that would be obtained at angles $\varnothing = 0^0, 45^0, 90^0, 135^0$, and 180^0

b) For the object shown in Figure 4, sketch the sinogram for values of $\varnothing$ from 0 to 180^0.

Exercise 3 – Reconstruction Methods

CT images can be reconstructed using different methods. In this exercise, we want to give you an insight into how these methods work by applying them to a simplified example; we investigate a matrix of only 2x2 elements. Don't worry; this exercise is only long, but not too tough!

a. Given is the 2x2 image below. The intensity values of the pixels are A11 to A22, while the projections along certain directions are named P0(i) and P90(i).



Use <u>direct matrix inversion</u> to calculate the values of A11 to A22.

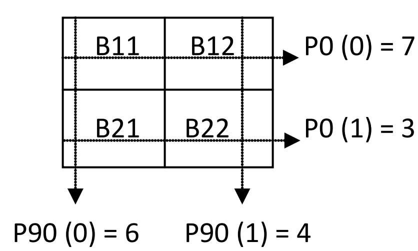
Hint:

You have a system of 4 unknowns and 4 possible equations. Write a matrix system $Fa = p$, where a and p are the pixel and projections written into a vector, and F is a 4x4 matrix filled with 0 and 1 to match the corresponding equations. Here are some matrices and their inversions which might be handy:

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow F^{-1} = \begin{pmatrix} 0 & -0.5 & 0.5 & 0.5 \\ 1 & 0.5 & -0.5 & -0.5 \\ 0 & 0.5 & 0.5 & -0.5 \\ 0 & 0.5 & -0.5 & 0.5 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow F^{-1} = \begin{pmatrix} 0 & 0.5 & 0.5 & -0.5 \\ 1 & -0.5 & -0.5 & 0.5 \\ 0 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 0.5 & 0.5 \end{pmatrix}$$

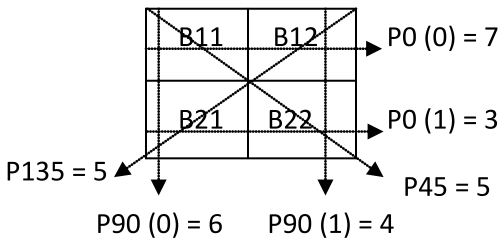
b. A similar image is given below. Use <u>the back projection theorem</u> to calculate the pixel values.



To do that, for each pixel, sum the values of all projections passing through it. Then rescale the resulting image by dividing it by $\frac{\text{the sum of all the new projections values}}{\text{the sum of all the original projections values}}$. Compare your result to $b_2 = F^T p$.

Why can't you compute F^{-1} and what does it imply on the reconstructed image using only these projections ?

c. The same set of pixels as in b is given, but now with two extra projections.



The number of equations is larger than the number of unknowns. In general, image reconstruction is a minimization problem. Here we want $b_{image} = \mathrm{argmin}_{\,\mathrm{b}} \|Fb - p\|^2$. The solution to this problem is the pseudo-inverse:

$$b_{image} = F_{PI}\, p \; = (F^T F)^{-1} F^T p$$

Compute b_{image} (using the hint).

Compute the new back-projected image, and compute the residual value r with b_{image} :

$$r = \left\| b_{image} - b_{backproj} \right\|^2$$

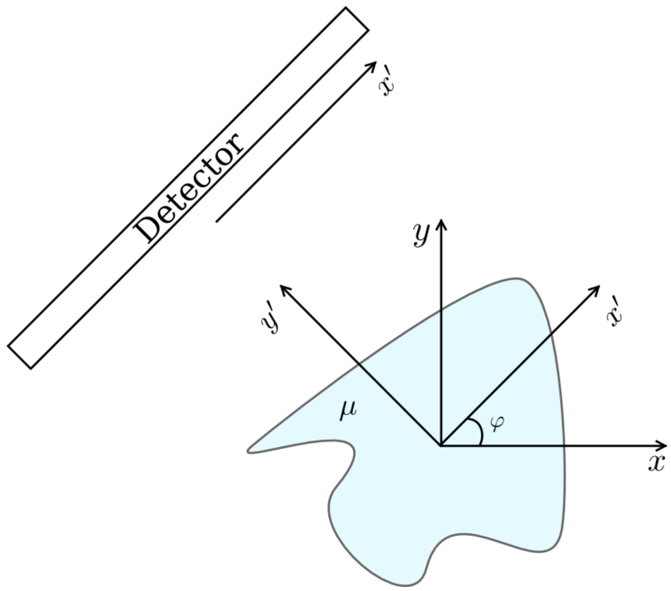
How does it compare to the residual value of point b. ? What do you conclude about back-projection ?

Hint:

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow F_{PI} = \frac{1}{6} \begin{pmatrix} 2 & -1 & 2 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 & -1 & 2 \\ -1 & 2 & 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & 2 & 2 & -1 \end{pmatrix}$$

Exercise 4 – Spatial Resolution of Back-Projection

You saw in the course that one of the issues applying back projection is the low spatial resolution. Here, we try to investigate this mathematically. We are interested in the attenuation coefficient as a function of position $\mu(x, y)$. We measure $g(x', \varphi)$, the radon transform of the μ along y' (see picture).



1. Justify why the back-projected $\mu_B(x, y)$ can be written as

$$\mu_B(x, y) = \int_{-\infty}^{+\infty} \int_{0}^{\pi} g(x', \varphi)\delta(x' - (\cos(\varphi)\, x + \sin(\varphi)\, y))\, d\varphi\, dx'$$

.

2. Write the relationship between $g(x', \varphi)$ and $\mu(x, y)$ of the tissue.
3. We will need to investigate rotation in k-space: show that if $\vec{r} = O\vec{r}'$, where O is a rotation matrix then the Fourier transform $\tilde{f}$ of a function f obeys $\tilde{f}(O\vec{k}) = \int f(O\vec{r}')e^{i\vec{k}\cdot\vec{r}'} d^2r'$.
4. Deduce from the previous results that

$$\tilde{\mu}(\cos(\varphi)\, k_x, \sin(\varphi)\, k_x) = \int g(x', \varphi)e^{ik_x x'} dx'$$

   Use this result to show $\mu_B(x, y) = \int \tilde{\mu}(k_x, k_y)e^{-i\vec{k}\cdot\vec{r}} \frac{d^2k}{|\vec{k}|}$, where the formally correct normalization of Fourier transforms has been omitted for simplicity.
   Hint: use the following relation: $\int e^{ik(x-d)} dk = \delta(x - d)$
5. Using symmetry and dimensionality arguments, show that the Fourier transform of $\frac{1}{|\vec{k}|}$ is proportional to $\frac{1}{|\vec{x}|}$.
6. Explain the blurring of the image back-projected image using these results, and describe the algorithm of filtered back-projection.