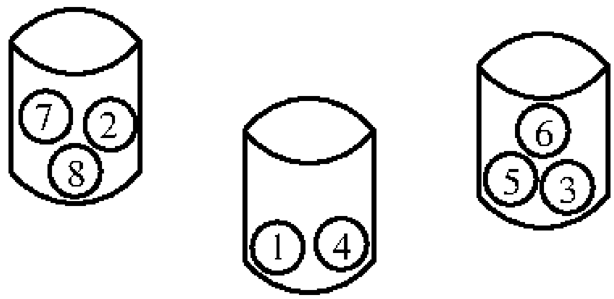
Homework 7: revision exercises (last year's midterm)

Exercise 1. There are $N \geq 2$ numbered balls divided into 3 urns:



We are interested in modelling the following process: at each time n, pick a number k uniformly at random in the set $\{1, \ldots, N\}$ (and independently from the previous picks); look then for the ball with this number, take it out of its urn and throw it uniformly at random in one of the other two urns.

We would like to keep track of the number of balls in one particular urn (say the first one) over time. Define X_n to be this number at time $n \geq 0$. The process $(X_n,\, n \geq 0)$ is then a Markov chain with state space $S = \{0, \ldots, N\}$.

a) Compute the transition matrix P of this chain.

b) Is the chain irreducible? aperiodic? Please justify.

c) For a given state $i \in S$, define $T_i = \inf\{n \geq 1 : X_n = i\}$. What are the values of

$$f_{00} = \mathbb{P}(T_0 < +\infty \,|\, X_0 = 0) \quad \text{and} \quad f_{0N} = \mathbb{P}(T_N < +\infty \,|\, X_0 = 0) \quad ?$$

d) Without computing it, explain why the chain admits a unique stationary distribution π.

e) Compute π. Are the detailed balance equations satisfied?

Hint: You should remember here Newton's binomial formula, valid for any $a, b \in \mathbb{R}$:

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k\, b^{N-k}, \quad \text{where } \binom{N}{k} = \frac{N!}{k!\,(N-k)!}$$

f) What is the value of $\mathbb{E}(T_0 \,|\, X_0 = 0)$? (with T_0 defined as in part c)

g) Do $\lim_{n\to\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0)$ and $\lim_{n\to\infty} \mathbb{P}(X_n = N \,|\, X_0 = 0)$ exist? If yes, what are the values of these two limits? If no, explain why these two limits do not exist.

BONUS: h) What is the value of $\sum_{i \in S} i\, \pi_i$?

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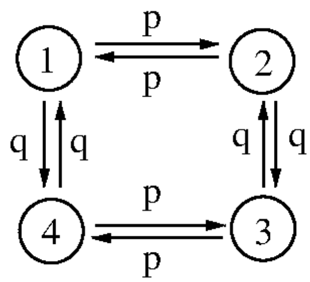
Exercise 2. Let $0 \leq p \leq 1$ and $q = 1 - p$. We consider the Markov chain with the following transition graph:



a) For what values of $0 \leq p \leq 1$ is the chain irreducible? Please justify.

b) For what values of $0 \leq p \leq 1$ is the chain aperiodic? Please justify.

c) For what values of $0 \leq p \leq 1$ does the chain admit a *unique* stationary distribution π? Compute π for these values.

d) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ of the transition matrix P of the chain (as functions of the parameter p).

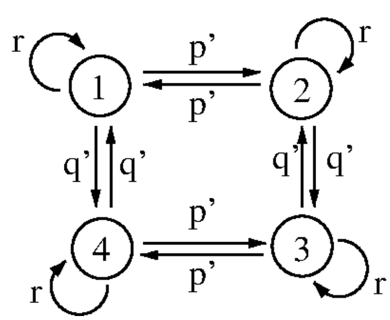
Hints: - You know already two of them!

- Reminder for computing the determinant of an $n \times n$ matrix A: for any $1 \leq i \leq n$, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \, a_{ij} \, \det(A(i,j))$$

where $A(i,j)$ denotes the $(n-1) \times (n-1)$ matrix A with deleted row i and column j.

Consider now the Markov chain with added self-loops:



where $0 < r < 1$, $p' = (1-r)\,p$ and $q' = (1-r)\,q$ [so that $p' + q' + r = 1$].

e) Compute the eigenvalues $\lambda'_0 \geq \lambda'_1 \geq \lambda'_2 \geq \lambda'_3$ of this new chain (as functions of the parameters p and r).

f) For any given value of $0 < p < 1$, find the value of $r(p)$ that maximizes the spectral gap of this new chain.

g) For what value of the triple (p', q', r) is the spectral gap the largest as possible?