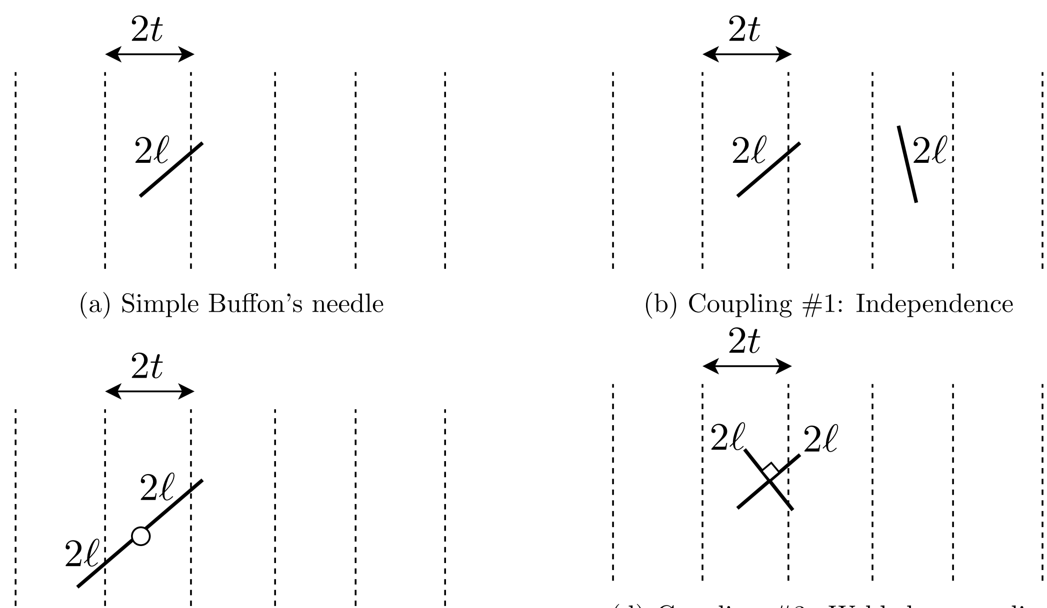
Homework 4 (due Friday, October 18)

Exercise 1. (Buffon's needle) We throw a needle of length 2ℓ *"randomly"* on a floor on which are already drawn parallel vertical lines, separated by equal distance $2t$ - see Fig. 1a. This needle is called Buffon's needle. Now, consider two Buffon's needles with $\ell = t$. We know for a fact that the probability with which a single needle crosses one of the vertical lines is equal to $\frac{2}{\pi}$. Generally, we do not know anything about the relation of these two needles to each other. However, by adding some assumptions, one can create a coupling between the two needles. For each of the following couplings, compute the probability with which "both" needles cross a vertical line.

Coupling #1: The needles are independently thrown on the floor - Fig. 1b.

Coupling #2: The needles are welded at their ends to form a straight needle with length $4\ell = 4t$ - Fig. 1c.

Coupling #3: The needles are welded perpendicularly at their midpoints, yielding a cross - Fig. 1d.



(a) Simple Buffon's needle

(b) Coupling #1: Independence

(c) Coupling #2: Welded at the end

(d) Coupling #3: Welded perpendicularly at midpoints

Figure 1: The same cup of coffee. Two times.

Exercise 2. A complete graph K_m with m vertices is a graph with one edge between *all* pairs of vertices. We consider the homogeneous random walk $(X_n,\, n \geq 0)$ on K_m defined by the transition probability $\mathbb{P}(X_{n+1} = j | X_n = i) = 1/(m-1)$ for all vertices $j \neq i$. We also denote by $\mu_i^{(n)} = \mathbb{P}(X_n = i)$, $i \in K_m$ the probability distribution of a random walk at time n with initial condition $X_0 = i_0$ and by $\nu_j^{(n)} = \mathbb{P}(Y_n = j)$, $j \in K_m$ the probability distribution of another independent random walk with initial condition $Y_0 = j_0$. The two initial vertices are fixed once for all and *distinct*, $i_0 \neq j_0$.

We now define the following homogeneous Markov process $((X_n, Y_n), n \geq 0)$ with state space $K_m \times K_m$ and transition probabilities

$$\mathbb{P}(X_{n+1} = i', Y_{n+1} = j' | X_n = i, Y_n = j) = \begin{cases} \frac{1}{(m-1)^2} & \text{if } i \neq j \text{ and } i' \neq i, j' \neq j, \\ \frac{1}{m-1} & \text{if } j = i \text{ and } j' = i' \neq i, \\ 0 & \text{in all other cases.} \end{cases}$$

It is understood that this Markov process is conditioned on the initial condition $(X_0, Y_0) = (i_0, j_0)$.

a) Show that (X_n, Y_n) is a coupling of the two probability distributions $\mu^{(n)}$ and $\nu^{(n)}$.

Hint: Recall the definition of coupling; you have to compute the marginals of $\mathbb{P}(X_n = i, Y_n = j)$.

b) Consider the coalescence time $T = \inf\{n \geq 1 \mid X_n = Y_n\}$ (a random variable). First show that for all $n \geq 1$:

$$\mathbb{P}(T > n) = \left(1 - \frac{m-2}{(m-1)^2}\right)^n$$

and deduce from there that for all $n \geq 1$:

$$\mathbb{P}(T = n) = \frac{m-2}{(m-1)^2} \left(1 - \frac{m-2}{(m-1)^2}\right)^{n-1}$$

Hint: rewrite the events $\{T > n\}$ and $\{T = n\}$ in terms of the X's and Y's.

c) Consider the total variation distance $\|\mu^{(n)} - \nu^{(n)}\|_{\mathrm{TV}}$. Use the formula in point 1 of exercise 2 to show that

$$\|\mu^{(n)} - \nu^{(n)}\|_{\mathrm{TV}} \leq e^{-n(m-2)/(m-1)^2}$$

Hint: $1 - x \leq e^{-x}$ for all $x \geq 0$.

d) We remark that $\pi_i = 1/m$, $i \in K_m$ is a stationary distribution for the random walk on the complete graph. Justify this remark. Use this remark and the previous bound to deduce

$$\|\mu^{(n)} - \pi\|_{\mathrm{TV}} \leq e^{-n(m-2)/(m-1)^2}$$

e) What happens in the (very) particular case $m = 2$?

Exercise 3. **a)** Let μ and ν be two distributions on a state space S (i.e., $\mu_i, \nu_i \geq 0$ for every $i \in S$ and $\sum_{i \in S} \mu_i = \sum_{i \in S} \nu_i = 1$). Show that the following three definitions of the total variation distance between μ and ν are equivalent:

1. $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i|$.
2. $\|\mu - \nu\|_{\mathrm{TV}} = \max_{A \subset S} |\mu(A) - \nu(A)|$, where $\mu(A) = \sum_{i \in A} \mu_i$ and $\nu(A) = \sum_{i \in A} \nu_i$.
3. $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \max_{\phi: S \to [-1, +1]} |\mu(\phi) - \nu(\phi)|$, where $\mu(\phi) = \sum_{i \in S} \mu_i \phi_i$ and $\nu(\phi) = \sum_{i \in S} \nu_i \phi_i$.

Hint: The easiest way is to show that $1 \leq 2 \leq 3 \leq 1$.

b) Show that $\|\mu - \nu\|_{\mathrm{TV}}$ is indeed a distance (i.e., that it is non-negative, that it is zero if and only if $\mu = \nu$, that it is symmetric and that the triangle inequality is satisfied).