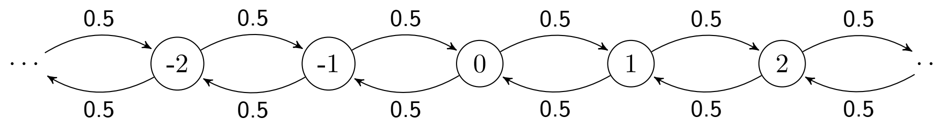
Random Walks: WEEK 6

1 Reflection principle

First, let us recall a few things about the symmetric random walk on $\mathbb{Z}$. We denote by $(S_n, n \geq 0)$ the simple symmetric random walk on $\mathbb{Z}$.



We know that this chain is irreducible and recurrent, that is:

$$f_{00} = \mathbb{P}(T_0 < +\infty | S_0 = 0) = 1$$

More than that, we know that it is null-recurrent, meaning:

$$\mathbb{E}(T_0 | S_0 = 0) = +\infty$$

How could we compute $f_{00}(2n) = \mathbb{P}(T_0 = 2n | S_0 = 0)$?

Due to the fact that we can translate the random walk along $\mathbb{Z}$ without actually changing anything, we will use the following notation:

$$p_{j-i}(n) = p_{ij}(n) = \mathbb{P}(S_n = j | S_0 = i)$$

The reflection principle is the proof technique that we are going to use to prove the following statement.

Theorem 1.1. Let $T_0 = \inf\{n \geq 1 : S_n = 0\}$. Then $\mathbb{P}(T_0 > 2n | S_0 = 0) = p_0(2n)$.

The left-hand side of the equality is the probability of never coming back to 0 before $2n$ steps. The right-hand side is the probability of being at 0 at time $2n$. From this theorem, we can then compute $f_{00}(2n)$ (left as an exercise).

Proof. First of all, let us shift the starting index by using the symmetry of the random walk.

$$\begin{aligned}
\mathbb{P}(T_0 > 2n | S_0 = 0) &= \mathbb{P}(T_0 > 2n, S_1 = +1 | S_0 = 0) + \mathbb{P}(T_0 > 2n, S_1 = -1 | S_0 = 0) \\
&= \mathbb{P}(T_0 > 2n | S_1 = +1, S_0 = 0)\, \mathbb{P}(S_1 = +1 | S_0 = 0) + \\
&\quad\ \mathbb{P}(T_0 > 2n | S_1 = -1, S_0 = 0)\, \mathbb{P}(S_1 = -1 | S_0 = 0) \\
&= \frac{1}{2}\mathbb{P}(T_0 > 2n | S_1 = +1) + \frac{1}{2}\mathbb{P}(T_0 > 2n | S_1 = -1)
\end{aligned}$$

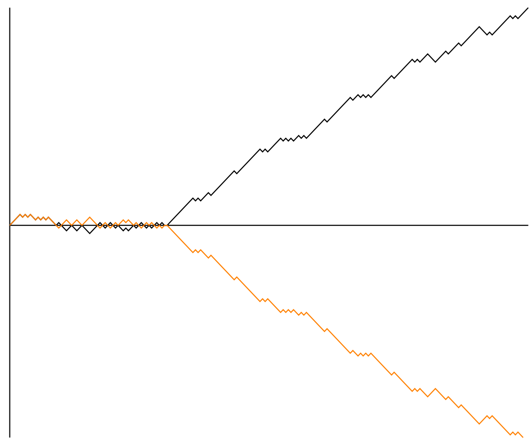
By symmetry, $\mathbb{P}(T_0 > 2n | S_1 = +1) = \mathbb{P}(T_0 > 2n | S_1 = -1)$ and we get:

$$\mathbb{P}(T_0 > 2n | S_0 = 0) = \mathbb{P}(T_0 > 2n | S_1 = 1)$$

Let us now distinguish paths depending on the point they are at time $2n$. Note that it has to be positive and even.

$$\begin{aligned}
\mathbb{P}(T_0 > 2n | S_1 = 1) &= \mathbb{P}(S_2 \neq 0, \ldots, S_{2n} \neq 0 | S_1 = 1) \\
&= \sum_{k \geq 1} \Big\{ \mathbb{P}(S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 2k | S_1 = 1) \Big\} \\
&= \sum_{k \geq 1} \Big\{ \mathbb{P}(S_{2n} = 2k | S_1 = 1) - \mathbb{P}(S_{2n} = 2k, \exists 2 \leq m \leq 2n-1 : S_m = 0 | S_1 = 1) \Big\}
\end{aligned}$$

The first term in the sum is simply the probability to be at $2k$ after $2n-1$ steps starting from 1. The second term is the probability to be at $2k$ after $2n-1$ steps starting from 1, but after hitting the 0-axis at some point. For any such path, we can draw a "mirror" path, as shown in the following graph, that will end up to be at $-2k$ after $2n-1$ steps.



The mirror path coincides with the original path until it hits zero. Then, it is a mirrored version of the original path.

Since any path that starts from 1 and ends up in $-2k$ has to cross the 0-axis, this shows that the number of paths described in the second term of the sum is exactly the number of paths that start from 1 and end in $-2k$ after $2n-1$ steps.

From this, we can further simplify the sum:

$$\begin{aligned}
\mathbb{P}(T_0 > 2n | S_1 = 1) &= \sum_{k \geq 1} \Big\{ \mathbb{P}(S_{2n} = 2k | S_1 = 1) - \mathbb{P}(S_{2n} = -2k | S_1 = 1) \Big\} \\
&= \sum_{k \geq 1} \Big\{ p_{2k-1}(2n-1) - p_{2k+1}(2n-1) \Big\}
\end{aligned}$$

This is a telescopic sum whose terms are null after some index, because $p_{2k+1}(2n-1) = 0$ for $k \geq n$. Therefore, the only remaining term is $p_1(2n-1)$ and we get:

$$\mathbb{P}(T_0 > 2n | S_1 = 1) = p_1(2n-1)$$

Finally, by the same argument of symmetry as in the beginning, we can conclude:

$$\mathbb{P}(T_0 > 2n | S_0 = 0) = p_0(2n)$$

$\square$

2 Consequences

2.1 Null-recurrence of the symmetric random walk on $\mathbb{Z}$

We now have a third proof of the fact that the simple symmetric random walk on $\mathbb{Z}$ is null-recurrent, i.e.

$$\mathbb{E}(T_0 | S_0 = 0) = +\infty$$

Proof. By using the lemma from last time:

$$\mathbb{E}(T_0|S_0=0) = \sum_{n\geq 1} \mathbb{P}(T_0 \geq n|S_0=0) \geq \sum_{n\geq 1} \mathbb{P}(T_0 > 2n|S_0=0)$$

Now we just apply the theorem:

$$\mathbb{E}(T_0|S_0=0) \geq \sum_{n\geq 1} p_0(2n)$$

Recall that $p_0(2n) \sim \frac{1}{\sqrt{\pi n}}$. Since $\sum_{n\geq 1} \frac{1}{\sqrt{\pi n}}$ diverges to $+\infty$, then $\sum_{n\geq 1} p_0(2n)$ also diverges to $+\infty$ and $\mathbb{E}(T_0|S_0=0) = +\infty$. □

2.2 The arcsine law

When we average everything, a symmetric random walk on $\mathbb{Z}$ will spend half its time above the 0-axis and half its time below. But what will actually typically happen is that the random walk will either spend most of its time above the 0-axis or most of its time below. We express this with the following theorem.

Definition 2.1. We define $L_{2n} = \sup\{0 \leq m \leq 2n : S_m = 0\}$ the time of last visit to 0 before $2n$.

Theorem 2.2. For n and k large (typically, $k = xn, 0 < x < 1$):

$$\mathbb{P}(L_{2n} = 2k|S_0=0) \sim \frac{1}{\pi\sqrt{k(n-k)}}$$

It means that a typical trajectory will cross 0 either at beginning or at the end.

Proof.

$$\begin{aligned}
\mathbb{P}(L_{2n} = 2k|S_0=0) &= \mathbb{P}(S_m \neq 0 \text{ for } m \in \{2k+1,\ldots,2n\}, S_{2k}=0|S_0=0) \\
&= \mathbb{P}(S_m \neq 0 \text{ for } m \in \{2k+1,\ldots,2n\}|S_{2k}=0)\,\mathbb{P}(S_{2k}=0|S_0=0) \\
&= \mathbb{P}(S_m \neq 0 \text{ for } m \in 1,\ldots,2(n-k)|S_0=0)\,p_0(2k) \\
&= p_0(2(n-k))\,p_0(2k) \text{ (by the theorem)} \\
&\sim \frac{1}{\sqrt{\pi(n-k)}}\frac{1}{\sqrt{\pi k}} \\
&\sim \frac{1}{\pi\sqrt{k(n-k)}}
\end{aligned}$$

□

2.3 Law of the iterated logarithm

We will not prove it, but we can also get the following result:

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = +1\right) = \mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1\right) = 1$$

This provides an envelope that the random walk will almost surely keep hitting.

3 Reversible chains

The ergodic theorem provides us with a nice convergence result, that is $\lim_{n\to\infty} p_{ij}(n) = \pi_j^*$ for any $i, j \in S$. But for the purpose of any practical application, we would like to know more about the rate at which this convergence occurs. We will start by talking about reversible chains and detailed balanced.

Definition 3.1. An ergodic Markov chain $(X_n, n \geq 0)$ is said to be reversible if its stationary distribution π^* satisfies the following detailed balance equation:

$$\pi_i^* p_{ij} = \pi_j^* p_{ji} \quad \forall i, j \in S$$

Remarks.

- We can still talk about reversibility if the chain is only irreducible and positive-recurrent.
- If one assumes a that the chain is in stationary distribution from the start then the backwards chain $X_n, X_{n-1}, \ldots$ has the same transition probabilities as the original chain, hence the name "reversible".
- If π^* satisfies the detailed balance equation, then $\pi^* = \pi^* P$.
- The reciprocal statement is wrong, as we will see in some counter-examples.
- We do not have general conditions that ensure that the detailed balance equation is satisfied.

Example 3.2 (Ehrenfest urns process)**.** Consider 2 urns with N numbered balls. At each step, we pick uniformly at random a number between 1 and N, take the ball with this number and put it in the other urn. The state is the number of balls in the right urn. The transition probabilities are the following:

$$\begin{aligned} p_{i,i+1} &= \frac{N-i}{N} \\ p_{i,i-1} &= \frac{i}{N} \end{aligned}$$

If we try to solve the detailed balance equation we get:

$$\begin{aligned} & \pi_{i+1}^* = \frac{p_{i,i+1}}{p_{i+1,i}} \pi_i^* = \frac{N-i}{i+1} \pi_i^* \\ \Rightarrow \quad & \pi_{i+1}^* = \frac{(N-i)(N-i-1)\ldots N}{(i+1)i(i-1)\ldots 2} \pi_0^* = \frac{N!}{(N-i-1)!(i+1)!} \pi_0^* \\ \Rightarrow \quad & \pi_{i+1}* = \binom{N}{i+1} \pi_0^* \end{aligned}$$

which leads to the conclusion that $\pi_0^* = \frac{1}{2^N}$ (see Homework 4),This process is therefore reversible.

Example 3.3. All irreducible birth-death processes (as also studied in Homework 4) satisfy the detailed balance equation.

Example 3.4. If for any $i, j \in S$, we have $p_{ij} > 0$ and $p_{ji} = 0$, then the chain is not reversible.

Example 3.5 (Random walk on the circle)**.** We know that the stationary distribution for the cyclic random walk on a circle with transition probabilities p and q ($p+q = 1$) is simply the uniform distribution, $\pi_i^* = \frac{1}{N}$. To be verified, the detailed balance equation requires $\pi_i^* p = \pi_{i+1}^* q \Leftrightarrow p = q = \frac{1}{2}$, which is not the case in general.