Homework 8 (due Friday, November 22)

Exercise 1. [Barker's algorithm]
Let $\pi = (\pi_i,\, i \in S)$ be a distribution on a finite state space S such that $\pi_i > 0$ for all $i \in S$ and let us consider the base chain with transition probabilities ψ_{ij}, which is assumed to be irreducible, aperiodic and such that $\psi_{ij} > 0$ if and only if $\psi_{ji} > 0$. Define the following acceptance probabilities:

$$a_{ij} = \frac{\pi_j\, \psi_{ji}}{\pi_i\, \psi_{ij} + \pi_j\, \psi_{ji}}$$

as well as a new chain with transition probabilities $p_{ij} = \psi_{ij}\, a_{ij}$ if $j \neq i$. Show that this new chain is ergodic and that it satisfies the detailed balance equation:

$$\pi_i\, p_{ij} = \pi_j\, p_{ji}, \quad \forall i, j \in S$$

Exercise 2. [Metropolized independent sampling in a particular case]
Let $0 < \theta < 1$ and let us consider the following distribution π on $S = \{1, \ldots, N\}$:

$$\pi_i = \frac{1}{Z}\, \theta^{i-1}, \quad i = 1, \ldots, N$$

where Z is the normalization constant, whose computation is left to the reader.

a) Consider the base chain $\psi_{ij} = \frac{1}{N}$ for all $i, j \in S$ and derive the transition probabilities p_{ij} obtained with the Metropolis-Hastings algorithm.

b) Using the result of the course, derive an upper bound on $\|P_i^n - \pi\|_{\mathrm{TV}}$. Compare the bounds obtained for $i = 1$ and $i = N$ (for large values of N).

c) Deduce an upper bound on the (order of magnitude of the) mixing time

$$T_\varepsilon = \inf\{n \geq 1 \,:\, \max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \varepsilon\}$$

Exercise 3. [Coupling]
The first goal of this exercise is to show that for any two distributions μ and ν on a common state space S, there always exist two (coupled) random variables X and Y with values in S such that

$$\mathbb{P}(X = i) = \mu_i,\ \forall i \in S \qquad \mathbb{P}(Y = j) = \nu_j,\ \forall j \in S \qquad \text{and} \qquad \|\mu - \nu\|_{\mathrm{TV}} = \mathbb{P}(X \neq Y)$$

Remember that in general, if X and Y satisfy the first two conditions, then we only have an inequality in the third statement. We need therefore to find a proper joint distribution for X and Y such that equality holds.

a) Define first $\xi_i = \min(\mu_i, \nu_i)$ for $i \in S$. Note that ξ itself is *not* a distribution, as $\sum_{i \in S} \xi_i \leq 1$ in general. Show that setting $\mathbb{P}(X = Y = i) = \xi_i$ for all $i \in S$ implies indeed that

$$\|\mu - \nu\|_{\mathrm{TV}} = \mathbb{P}(X \neq Y)$$

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b) We need now to define $\mathbb{P}(X = i, Y = j)$ for $i \neq j$ so that $\mathbb{P}(X = i) = \mu_i$, $\forall i \in S$ and $\mathbb{P}(Y = j) = \nu_j$, $\forall j \in S$. Show that the following proposal works (it is not the unique one):

$$\mathbb{P}(X = i, Y = j) = \frac{(\mu_i - \xi_i)\,(\nu_j - \xi_j)}{1 - \sum_{k \in S} \xi_k}$$

[In particular, observe that there are lots of zeros in this joint distribution: if $\mu_i \leq \nu_i$ for a given $i \in S$, then $\mathbb{P}(X = i, Y = j) = 0$ for all $j \in S \backslash i$; likewise, if $\nu_j \leq \mu_j$ for a given $j \in S$, then $\mathbb{P}(X = i, Y = j) = 0$ for all $i \in S \backslash j$. X and Y are therefore tightly coupled!]

NB: And what if $\sum_{k \in S} \xi_k = 1$?

c) Use this to show that for an ergodic Markov chain with transition matrix P and stationary distribution π, the total variation distance $d(n) = \max_{i \in S} \|P_i^n - \pi\|_{TV}$ is a non-increasing function of n.

Hint: A new coupling is required here.