Homework 9 (due Friday, November 29)

Exercise 1. a) Preliminary question. Consider the random walk on $\mathbb{Z}$ with transition probabilities $\psi_{i,i\pm 1} = 1/2$. Does this chain admit a stationary distribution ? Is it positive-recurrent?

In this problem, we use the Metropolis-Hastings rule to bias the simple random walk so that the stationary distribution of the new walk $(X_n, n \geq 0)$ equals

$$\pi_i = \frac{e^{-ai^2}}{\sum_{i=-\infty}^{+\infty} e^{-ai^2}}$$

where $a > 0$ is a parameter. Moves $i \to j = i \pm 1$ are proposed with probability ψ_{ij} and accepted with probability $a_{ij} = \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right)$.

b) Give the transition probabilities $p_{ij} = \mathbb{P}(X_1 = j \mid X_0 = i)$ of the final chain for *all* i and j.

c) Show that for this chain:

$$\lim_{n \to +\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j, \quad \forall i \in \mathbb{Z}.$$

For the next question, we consider two coupled walks $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ on $\mathbb{Z}$. The walks are coupled in the following way:

- At each time step $n \geq 0$, we draw a common uniform random variable $\xi_n \in \{+1, -1\}$ and for each walk we propose the moves $X_n \to X_n + \xi_n$ and $Y_n \to Y_n + \xi_n$.
- Each move is accepted or rejected according to the Metropolis-Hastings rule of question b).

We define the coalescence time (a random variable)

$$T = \inf\{n : X_n = Y_n \text{ given that } X_0 = z, Y_0 = z + d\}$$

where z and d are strictly *positive* integers.

d) What is the smallest possible coalescence time ? Compute the probability that the coalescence time takes this smallest possible value.