# SOLUTION SUGGESTIONS SÉRIE 2

**Solution** (Exercise 1)**.** Let $G$ be a group with p elements. Let

$$\mathcal{F}(G, A) = \{f : G \to A\}$$

For $h \in G$, $f \in \mathcal{F}(G, A)$, $h \star f \in \mathcal{F}(G, A)$ is defined by

$$h \star f(g) := f(gh)$$

called right translation of $f$ by $h$.

(1) (**Right translation is a right action**) For $h_1, h_2, g \in G$, $f \in \mathcal{F}(G, A)$, $(h_1h_2)\star f(g) = f(gh_1h_2) = h_2\star f(gh_1) = h_1\star(h_2\star f)(g)$ as desired.

Clearly $e_G \star f = f$.

(2) Let $h \in G$, $h \neq e_G$, $ord_G(h) \neq 1$. By Lagrange's theorem $ord_G(h) \mid |G|$, so $ord_G(h) = p$. (since $|G| = p$ is prime.) Therefore any element $h \neq e_G$ generates $G$, i.e. every element of $G$ is a power of $h$. We have $h \star f = f \implies g \star f = f \; \forall \; g \in G$ in particular $f(g) = g \star f(e_G) = f(e_G) \; \forall \; g \in G$. In other words, f is constant.

(3) By the Orbit-Stabilizer thm and the theorem of Lagrange, every orbit has either p elements or 1 element. By the discussion in the previous point, the orbits with one element precisely correspond to constant functions. We have

$$|\mathcal{F}(G, A)| = p.|\text{set of orbits with p elements}|+1.|\text{set ofconstant functions}|$$

$$a^p = p.|\text{set of orbits with p elements}| + 1.a$$

$$\text{No. of orbits} = |\text{set of orbits with p elements}|+|\text{set ofconstant functions}| = \frac{a^p - a}{p}+a$$

(4) Fermat's little theorem follows since |set of orbits with p elements| is an integer. So $p|(a^p - a)$ for every positive integers $a$, and also for negative integers by replacing $a$ by $-a$.

**Solution** (Exercise 2)**.** See Série 1 Corrigé

**Solution** (Exercise 4)**.** Suppose $G$ acts transitively on a set $X$. Let $x \in X$ be s.t. the group $G_x$ acts transitively on $X - \{x\}$. We wish to show that every point in $X$ has the property that its stabiliser acts transitively on X minus the point. Let $y \in X$ be arbitrary and $h \in G$ be s.t. $h(x) = y$. We have $G_y = hG_xh^{-1}$ and suppose $z_1, z_2 \in X$ be different from $y$, let $w_i = h^{-1}z_i$ we have $w \neq x$ and by assumption $\exists$ $g \in G_x$ s.t. $gw_1 = w_2$. It follows that $g' = hgh^{-1} \in G_y$ and $g'(z_1) = z_2$ as desired (i.e.) (1) $\Leftrightarrow$ (2).

Now assume (2). Let $(x, y), (u, v) \in X \times X - \Delta X$ i.e. $x \neq y$ and $u \neq v$. Since the action of $G$ on $X$ is transitive, $\exists$ $g_1 \in G$ s.t. $g_1(x) = u$ and let $y' := g_1(y)$. We observe that $u \neq y'$ since if $g_1(x) = g_1(y)$ by applying $g_1^{-1}$ we may conclude $x = y$. By (2) $\exists$ $g_2 \in G_u$ s.t. $g_2(y') = v$. So we have $g_2g_1(x, y) = (u, v)$ i.e. (2) $\implies$ (3).

Let us assume (3), let $x \in X$ and $u, v \in X$ be arbitrary different from $x$, we have $(x, u), (x, v) \in X \times X - \Delta_X$. By (3) $\exists$ $g \in G$ s.t. $g(x, u) = (x, v)$, i.e. $g \in G_x$ and $g(u) = v$. We have proved (2).

**Solution** (Exercise 3)**.** We would like to present a solution of Exercise 3 along the lines of Exercise 4: Let $X := \mathbb{R}^2$ and $G = Isom(\mathbb{R}^2)$, we know that $G$ acts transitively on $X$.

The following statements are equivalent:

(1) $\exists$ $x \in X$ s.t. $G_x$ acts transitively up to preserving distances i.e. $\forall$ $y, z \in X$ s.t. $d(x, y) = d(x, z)$ $\exists$ $g \in G_x$ s.t. $gy = z$.
(2) $\forall$ $x \in X$, $G_x$ acts transitively up to preserving distances i.e. $\forall$ $x, y, z \in X$ s.t. $d(x, y) = d(x, z)$ $\exists$ $g \in G_x$ s.t. $gy = z$.
(3) The orbits of the action of $G$ on $X \times X$ are given by $(X \times X)_r := \{(x, y) | x, y \in X \text{ with } d(x, y) = r\}$ for every $r \geq 0$.

We will leave the proof of the above equivalence to the reader – it is similar to the proof of exercise 4. Let us use the above result to prove exercise 3. Using (1) $\implies$ (3) in order to show $G$ acts transitively on $(X \times X)_1$ it suffices to check $G_0$ (linear isometries) acts transitively on $X$ upto preserving distances. But this clear since $\forall$ $y, z \in X$ s.t. $d(0, y) = d(0, z)$ $\exists$ a rotation $g \in G_0$ s.t. $gy = z$.

**Solution** (Exercise 5)**.** We make a table with elements of $D_8$, their order and number of fixed points in $\mathcal{F}(4, c)$: Here $r$ is an order 4 rotation and $s$ is the reflection about a line through a pair of opposite sides. ($r^2s$ will then be reflection about a line through the perpendicular sides, $rs, r^3s$ will be reflection about line through pair of opposite vertices)

| Element | Order | No. of fixed points |
|---|---|---|
| Identity | 1 | $c^4$ |
| $r$ | 4 | $c$ |
| $r^2$ | 2 | $c^2$ |
| $r^3$ | 4 | $c$ |
| $s$ | 2 | $c^2$ |
| $rs$ | 2 | $c^3$ |
| $r^2s$ | 2 | $c^2$ |
| $r^3s$ | 2 | $c^3$ |

By Burnside's formula, the number of possible necklaces:

$$|D_8 \backslash \mathcal{F}(4, c)| = \frac{1}{8}(c^4 + 2c^3 + 3c^2 + 2c)$$

The number of orbits by Burnside formula is :

$$|C_4 \backslash \mathcal{F}(4, c)| = \frac{1}{4}(c^4 + c^2 + 2c)$$

*Remark.* Note that conjugate elements have same order and number of fixed points. Being conjugate is an equivalence relation and the equivalence classes are called conjugacy classes.

**Solution.** We are interested in the number of colorings of a regular pentagon, we tabulate according to the conjugacy classes:

| Class | No.of elts | Order | No. of fixed points |
|---|---|---|---|
| Identity | 1 | 1 | $c^5$ |
| $\{r^2, r^{-2}\}$ | 2 | 5 | $c$ |
| $\{r, r^{-1}\}$ | 2 | 5 | $c$ |
| Reflections | 5 | 2 | $c^3$ |

By Burnside formula, the number of necklaces is

$$\frac{1}{10}(c^5 + 5c^3 + 4c)$$

□

**Solution** (Exercise 7)**.** We may view the carbon atoms as forming a regular hexagon and each molecule as a coloring of this hexagon by two colors – namely chlorine and hydrogen. So the problem is to count the number of distinct colourings of a regular hexagon with 2 colours. As we have seen in the course, we can label the hexagon and view each coloring as a map from the set of labels to the colors. The group of isometries of the hexagon (isomorphic to the dihedral group $D_{12}$) acts on the labels and therefore on the labeled colourings of the hexagon. The distinct colourings will correspond to distinct orbits under this action. The strategy is to count the number of orbits using Burnside formula. We have tabulated the various classes of elements, the number of elements of each class, the order and the number of fixed points. Note that the order and number of fixed points will depend only on the class of the element – since any two elements of the same class are conjugate to each other by an element of $D_{12}$. Let $r$ denote the generator of the subgroup of rotations of $D_{12}$.

| Class | No.of elts | Order | No. of fixed points |
|---|---|---|---|
| Identity | 1 | 1 | $2^6 = 64$ |
| $\{r^3\}$ | 1 | 2 | $2^3 = 8$ |
| $\{r^2, r^{-2}\}$ | 2 | 3 | $2^2 = 4$ |
| $\{r, r^{-1}\}$ | 2 | 6 | 2 |
| Reflection through a pair of opposite vertices | 3 | 2 | $2^4 = 16$ |
| Reflection through a pair of opposite sides | 3 | 2 | $2^3 = 8$ |

By Burnside formula, the number of molecules is

$$\frac{1}{12}(1.64 + 1.8 + 2.4 + 2.2 + 3.16 + 3.8) = \frac{156}{12} = 13$$

□

**Solution** (Exercise 8)**.** Let $\phi : X \to Y$ be a morphism of $G - sets$ which is bijective.

$$\forall x \in X, g \in G, \phi(gx) = g\phi(x) \implies \forall y \in Y, g \in G, g\phi^{-1}(y) = \phi^{-1}(gy)$$

**Solution** (Exercise 9)**.** Let $\phi : X \to Y$ be a morphism of G-sets.

We want to show there is a unique map $\overline{\phi} : G\backslash X \to G\backslash Y$ defined by $\overline{\phi}(G.x) = G.y$. This map is well defined because $G.x_1 = G.x_2$ implies $\exists\ g \in G$ s.t. $x_2 = gx_1$, we have $\phi(x_2) = \phi(g.x_1) = g\phi(x_1)$ (the last equality is because $\phi$ is a morphism of G-sets) i.e. $G\phi(x_1) = G\phi(x_2)$. Uniqueness is clear.

Observe the following $id_X : X \to X$ the identity map is a morphism of G-sets and $\overline{id_X} = id_{G\backslash X}$. Let $\phi_1 : X \to Y$ and $\phi_2 : Y \to Z$ be morphism of G-sets. We have $\phi_2 \circ \phi_1 : X \to Z$ is a morphism of G-sets and $\overline{\phi_2 \circ \phi_1} = \overline{\phi_2} \circ \overline{\phi_1}$. It follows therefore that if $\phi : X \to Y$ is an isomorphism of G-sets, we have $\overline{\phi^{-1}} = \overline{\phi}^{-1}$ and so $\overline{\phi}$ is a bijection.

*Remark.* Fix a group $G$ and consider the class of $G - sets$. The correspondence from $G - sets$ to $sets$ sending $X$ to $G\backslash X$ and $\phi \in Hom_{G-sets}(X, Y)$ to $\overline{\phi} \in Hom(G\backslash X, G\backslash Y)$ is a *functor*.

**Solution** (Exercise 10)**.** Let $X$ be a G-set and $x \in X$, let $Gx$ be the orbit of $x$ and $G_x$ the stabilizer of $x$ under the action of G. The Orbit-Stabilizer theorem says the map

$$gG_x \in G/G_x \mapsto gx \in Gx$$

is a well defined map and moreover a bijection.

We want to observe that in fact $G/G_x$ is a G-set the action being defined by

$$h.(gG_x) := (hg)G_x$$

for $h \in G$. The reader is left to check that this is a well defined left action of $G$. Likewise $Gx$ is a G-set with the action defined by

$$h.(y) = hy$$

for $h \in G$ and $y \in Gx$. (Note that if $y = gx$, $hy = (hg)x$, so in fact $hy \in Gx$.) Again the reader should check that this is a left action. (This is clear because it is the restriction of the action on $X$ to $Gx$.) With these definitions of actions, it is clear that the bijection defined above is in fact a morphism of G-sets.