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Geometrie, MATH-125

Série 9

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1 Isometries lineaires

Exercice 1. Pour chacune des matrices suivantes determiner si elles sont orthogonales et calculer leur determinant.

$$\frac{1}{9}\begin{pmatrix} 8 & 1 & 4 \\ 1 & 8 & -4 \\ 4 & -4 & -7 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & -4 \\ 4 & -4 & -7 \end{pmatrix},$$

$$\frac{1}{3}\begin{pmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{pmatrix}, \quad \frac{1}{25}\begin{pmatrix} -9 & -12 & -20 \\ -20 & 15 & 0 \\ -12 & -16 & 15 \end{pmatrix}$$

Preuve: Denote the matrices by A.

1. For the first matrix, one can see that ${}^tAA = A{}^tA = \mathrm{Id}$, so it is an orthogonal matrix. Moreover $\det(A) = -1$, so it is a symmetric matrix.
2. For the second matrix, it is NOT an orthogonal matrix.
3. The third matrix is an orthogonal matrix and moreover $\det(A) = 1$.
4. The fourth matrix is an orthogonal matrix and $\det(A) = -1$.

Exercice 2.

Exercice 3. Soit φ une isometrie et $M = (x_{ij})_{i,j\leqslant n}$ sa matrice associee dans la base canonique. On rappelle que la trace de M est la somme des coefficient diagonaux

$$\mathrm{tr}(M) = \sum_{i=1}^{n} x_{ii}.$$

1. Montrer que
$$\sum_{i,j\leqslant n}\sum |x_{ij}|^2 = n$$
(utiliser le fait que M est orthogonale).

2. En deduire (utiliser Cauchy-Schwarz) que $|\mathrm{tr}(M)| \leqslant n$.
3. Montrer que si $|\mathrm{tr}(M)| = n$ alors $M = \pm \mathrm{Id}_n$.

Preuve:

1. Since M is an orthogonal matrix, we know each row vector of M is of norm 1 : for each $1 \leqslant i \leqslant n$, $\sqrt{\sum_{j=1}^n |x_{ij}|^2} = 1$. That is, $\sum_{j=1}^n |x_{ij}|^2 = 1$, for $1 \leqslant i \leqslant n$. Then, we sum over $1 \leqslant i \leqslant n$ to get $\sum_{i,j \leqslant n}\sum |x_{ij}|^2 = n$.
2. By Cauchy–Schwarz inequality, we have
$$|\mathrm{tr}(M)| = |\sum_{i=1}^n x_{ii}| \leqslant \left(\sum_{i=1}^n |x_{ii}|^2\right)^{1/2} \left(\sum_{i=1}^n 1\right)^{1/2} \leqslant \left(\sum_{i,j \leqslant n}\sum |x_{ij}|^2\right)^{1/2} \times \sqrt{n} = n.$$
3. We first note that since $\sum_{j=1}^n |x_{ij}|^2 = 1$, we have $|x_{ij}| \leqslant 1$, for any $1 \leqslant i, j \leqslant n$. If $|\mathrm{tr}(M)| = n$, then the inequalities in the proof of Part 2 should become equalities. In particular, for the second equality to hold, we must have $x_{ij} = 0$ whenever $i \neq j$. Then M becomes a diagonal matrix $\mathrm{diag}(x_{11}, x_{22}, \cdots, x_{nn})$. By further noticing that $|x_{ii}| \leqslant 1$ $(1 \leqslant i \leqslant n)$, $|\mathrm{tr}(M)| = n$ would imply $x_{ii} = 1$ for all $1 \leqslant i \leqslant n$, or $x_{ii} = -1$ for all $1 \leqslant i \leqslant n$. That is, we must have $M = \mathrm{Id}_n$ or $M = -\mathrm{Id}_n$.

Exercice 4.

Exercice 5. Soit $\mathbf{0} \neq \vec{v} \in \mathbb{R}^n$ un vecteur non-nul ; on rappelle que l'application

$$\varphi_{\vec{v}} : \vec{u} \in \mathbb{R}^n \mapsto \vec{u} - 2\frac{\langle \vec{u}, \vec{v}\rangle}{\langle \vec{v}, \vec{v}\rangle}\vec{v}$$

est une isometrie : la symetrie par rapport a l'hyperplan $\vec{v}^{\perp}$.

1. Montrer que $\vec{v}$ est un vecteur propre.
2. Montrer qu'il existe une base orthonormee formee uniquement de vecteurs propres de $\varphi_{\vec{v}}$, ie. une base orthonormee $(\mathbf{e}_i)_{i \leqslant n}$ telle que
$$\varphi(\mathbf{e}_i) = \lambda_i \mathbf{e}_i$$
(ne pas chercher tres loin).
3. Montrer que φ est non speciale.

Preuve:

1. $\varphi_{\vec{v}}(\vec{v}) = \vec{v} - 2\frac{\langle \vec{v}, \vec{v}\rangle}{\langle \vec{v}, \vec{v}\rangle}\vec{v} = -\vec{v}$. Therefore $\vec{v}$ is an eigenvector of $\varphi_{\vec{v}}$ with eigenvalue -1.

2. For any $\vec{u} \in \vec{v}^{\perp}$, we have $\varphi_{\vec{v}}(\vec{u}) = \vec{u}$. Then any $\vec{u} \in \vec{v}^{\perp}$ is an eigenvector of $\varphi_{\vec{v}}$ with eigenvalue 1. Let $\{\vec{u}_1, \cdots, \vec{u}_{n-1}\}$ be an orthonormal basis of $\vec{v}^{\perp}$. Then

$$\varphi_{\vec{v}}\left(\vec{u}_1, \cdots, \vec{u}_{n-1}, \frac{\vec{v}}{\|\vec{v}\|}\right) = \left(\vec{u}_1, \cdots, \vec{u}_{n-1}, \frac{\vec{v}}{\|\vec{v}\|}\right) \begin{pmatrix} 1 & & & \\ & \cdots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Then $(\mathbf{e}_i)_{i \leqslant n} := \{\vec{u}_1, \cdots, \vec{u}_{n-1}, \frac{\vec{v}}{\|\vec{v}\|}\}$ is an orthonormal basis satisfying the condition.

3. From part 2 we know that matrix M_{φ} corresponding to $(\mathbf{e}_i)_{i \leqslant n}$ is of determinant -1. In particular φ is non-special.

Exercice 6.

2 Isometries de $\mathbb{R}^3$

Exercice 7. (Criteres matriciels pour reconnaitre une isometrie de $\mathbb{R}^3$.) On connait la forme de la matrice d'une isometrie φ dans une BO convenable, mais souvent ce dont on dispose c'est de la matrice $M_{0,\varphi}$ de l'isometrie φ dans la base canonique. Dans cet exercice on explicite des criteres donnant des indices sur la nature de φ a partir de la matrice $M_{0,\varphi}$.

1. Montrer que si φ est une rotation, sa trace appartient a l'intervalle $[-1, 3]$. Que dire si sa trace vaut 3 ? si elle vaut -1 ?
2. Montrer que si φ est une anti-rotation, sa trace appartient a l'intervalle $[-3, 1]$. Que dire si sa trace vaut -3 ? si elle vaut 1 ?
3. Que vaut $\mathrm{tr}(\varphi)$ is φ est une symetrie (par rapport a un plan).
4. Soit $M = M_{0,\varphi}$ la matrice de φ dans la base canonique. Montrer que φ est l'identite ou bien une symetrie (par rapport a un plan, une droite ou encore a l'origine) si et seulement si est une matrice symetrique : ie.

$$^tM = M.$$

Pour cela on considerera la matrice de φ dans une base orthonormee convenable et on observera que la matrice de changement de base est elle aussi une matrice orthogonale.

5. Montrer que si φ est une symetrie, son type (identite, centrale, axiale, par rapport a un plan) est completement determine par sa trace.

6. Montrer que si M est une rotation ou une anti-rotation (d'axe $\mathbb{R}\mathbf{e}_1$ et d'angle $\mathrm{c}+i\mathrm{s}$ ou θ radians) on a

$$\mathrm{c} = \cos(\theta) = \frac{1}{2}(\mathrm{tr}M - \det(M)).$$

Cette formule permet donc de determiner $\pm\theta \pmod{2\pi}$ ou si on parle en terme de nombre complexes de module 1 de determine l'angle a conjugaison complexe pres :$\mathrm{c} \pm i\mathrm{s}$).

Preuve:

1. Under the convenient orthonormal basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the matrix of φ is of the form
$$M_{\varphi,\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix},$$
where $c = \cos(\theta)$ and $s = \sin(\theta)$. We know that the trace and determinant of φ do not depend on the choices of the bases (if $\mathcal{B}$ and $\mathcal{B}'$ are two bases and A is the base change matrix from $M_{\varphi,\mathcal{B}}$ to $M_{\varphi,\mathcal{B}'}$, then $M_{\varphi,\mathcal{B}'} = A \cdot M_{\varphi,\mathcal{B}} \cdot A^{-1}$ which would imply $\mathrm{tr}(M_{\varphi,\mathcal{B}'}) = \mathrm{tr}(M_{\varphi,\mathcal{B}})$ and $\det(M_{\varphi,\mathcal{B}'}) = \det(M_{\varphi,\mathcal{B}})$). Hence $\mathrm{tr}(\varphi) = \mathrm{tr}(M_{\varphi,\mathcal{B}}) = 1 + 2\cos(\theta) \in [-3, 3]$. If $\theta = \pi$, that is, if φ is an axial symmetry, then $\mathrm{tr}(\varphi) = 1 - 2 = -1$. If $\theta = 0$, i.e., $\varphi = id$, then $\mathrm{tr}(\varphi) = 3$.
2. Under the convenient orthonormal basis $\mathcal{B}$, the matrix of φ is of the form
$$M_{\varphi,\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, c = \cos(\theta), s = \sin(\theta).$$
Then $\mathrm{tr}(\varphi) = \mathrm{tr}(M_{\varphi,\mathcal{B}}) = -1 + 2\cos(\theta) \in [-3, 1]$. $\mathrm{tr}(\varphi) = -3$ if $\cos(\theta) = -1$, if $\theta = \pi$ and φ is a point symmetry (symetrie centrale). If $\mathrm{tr}(\varphi) = 1$, then $\cos(\theta) = 1$ and $\theta = 0$, in which case φ is an orthogonal symmetry with respect to the plane $\mathbb{R}\mathbf{e}_2 + \mathbb{R}\mathbf{e}_3$.
3. Under an appropriate orthonormal basis, the matrix of φ is of the form
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
which implies that $\mathrm{tr}(\varphi) = \mathrm{tr}(M) = 1$.
4. The matrix of a plane symmetry in the convenient orthonormal basis $\mathcal{B}$ is given by $M_{\varphi,\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let A be the base change matrix from $M_{\varphi,\mathcal{B}}$ to $M_{0,\varphi}$.

Then $M_{0,\varphi} = A \cdot M_{\varphi,\mathcal{B}} \cdot A^{-1} = A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA$, where A is an orthogonal matrix. Now

$$ {}^tM_{0,\varphi} = {}^t(A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA) = {}^t({}^tA) \cdot {}^tM_{\varphi,\mathcal{B}} \cdot {}^tA = A \cdot M_{\varphi,\mathcal{B}} \cdot {}^tA = M_{0,\varphi}, $$

since ${}^tM_{\varphi,\mathcal{B}} = M_{\varphi,\mathcal{B}}$.

5. (a) If $\varphi = id$ is the identity, then $\mathrm{tr}(\varphi) = \mathrm{tr}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = 3$.

   (b) If φ is a point symmetry (symetrie centrale), then from Part 2, we know that $\mathrm{tr}(\varphi) = \mathrm{tr}(M_{\varphi,\mathcal{B}}) = -1 = 2\cos(\pi) = -3$ (with $\theta = \pi$ there).

   (c) If φ is an axial symmetry, then from Part 1, we know that $\mathrm{tr}(\varphi) = \mathrm{tr}(M_{\varphi,\mathcal{B}}) = 1 + 2\cos(\pi) = -1$ (with $\theta = \pi$ there).

   (d) If φ is a symmetry with respect to a plane, then from Part 3, we know that $\mathrm{tr}(\varphi) = \mathrm{tr}(M) = 1$.

6. If φ is a rotation, then from Part 1, we know that under the basis $\mathcal{B}$, $\mathrm{tr}(\varphi) = 1+2\cos(\theta)$ and $\det(\varphi) = 1$. Then $\frac{1}{2}(\mathrm{tr}(M_{\varphi,\mathcal{B}})-\det(M_{\varphi,\mathcal{B}})) = \frac{1}{2}(1+2\cos(\theta)-1) = \cos(\theta)$.

   Similarly, if φ is an anti-rotation, then from Part 2, we know that $\mathrm{tr}(M_{\varphi,\mathcal{B}}) = -1 + 2\cos(\theta)$ and $\det(M_{\varphi,\mathcal{B}}) = -1$. Hence $\frac{1}{2}(\mathrm{tr}(M_{\varphi,\mathcal{B}}) - \det(M_{\varphi,\mathcal{B}})) = \frac{1}{2}(-1 + 2\cos(\theta) + 1) = \cos(\theta)$.

Exercice 8.

Exercice 9. Soit φ et ψ deux isometries affines de parties lineaires φ_0 et ψ_0.

1. Montrer que si φ est d'un certain type (translation, rotation, vissage, symetrie centrale, axiale, planaire, glissee, anti-rotation) alors la conjuguee

   $$ \varphi' = \mathrm{Ad}(\psi)(\varphi) = \psi \circ \varphi \circ \psi^{-1} $$

   est du meme type.
2. Montrer qu'en cas de rotation, anti-rotation ou vissage, l'angle est preserve au signe pres (si l'angle est le nombre complexe de module 1, z le nouvel angle sera $z^{\pm 1}$; ou si l'angle est exprime en radians $\theta \pmod{2\pi}$ le nouvel angle sera $\pm\theta \pmod{2\pi}$). Calculer l'axe de φ' en fonction de ψ et de l'axe de φ.
3. En general, quels sont les points fixes de φ' en fonction de ψ et de ceux de φ.

Preuve:

1. Let us consider first the case $\phi = t_u$ a translation. When $\psi = t_v$ is another translation, it is easy to check that

$$\mathrm{Ad}(\psi)(\phi) = t_v \circ t_u \circ t_v^{-1} = t_u \tag{2.1}$$

since translations commute which each other. When $\psi = \psi_0$ is linear we have

$$\mathrm{Ad}(\psi)(\phi) = \psi_0 \circ t_u \circ \psi_0^{-1} = t_{\psi_0(u)} \tag{2.2}$$

as one can see by checking what both members do to $z \in \mathbb{R}^3$. Therefore, writing ψ as a composition of a linear part ψ_0 and a translation t_v one proves that

$$\begin{aligned}\mathrm{Ad}(\psi)(\phi) =&(\mathrm{Ad}(t_v) \circ \mathrm{Ad}(\psi_0))(t_u) = \mathrm{Ad}(t_v)(\mathrm{Ad}(\psi_0)(t_u))\\ =&\mathrm{Ad}(t_v)(t_{\psi_0(u)}) = t_{\psi_0(u)}\end{aligned}$$

which is another translation (we applied successively (2.2) and (2.1)).
Now assume $\phi = \phi_0$ is linear and $\psi = t_v \circ \psi_0$ as before. In this case

$$\begin{aligned}\mathrm{Ad}(\psi)(\phi) =&t_v \circ \psi_0\phi_0\psi_0^{-1} \circ t_{-v} = t_v \circ t_{-\psi_0\phi_0\psi_0^{-1}(v)} \circ \psi_0\phi_0\psi_0^{-1}\\ =&t_{v-\psi_0\phi_0\psi_0^{-1}(v)} \circ \psi_0\phi_0\psi_0^{-1}\end{aligned} \tag{2.3}$$

by (2.2) again.
Let us observe that the linear part $\phi_0' = \psi_0\phi_0\psi_0^{-1}$ gets conjugated by ψ_0 (and therefore, has the same eigenvalues than ϕ_0) and there is a translation by a vector in the image of $\mathrm{Id} - \phi_0'$.
In the general case $\phi = t_v \circ t_w \circ \phi_0$ where $v \in \ker(\mathrm{Id} - \phi_0)$ and $w \in \mathrm{Im}(\mathrm{Id} - \phi_0)$. Then

$$\mathrm{Ad}(\psi)(\phi) = \mathrm{Ad}(\psi)(t_v \circ t_w \circ \phi_0) = \mathrm{Ad}(\psi)(t_v) \circ \mathrm{Ad}(\psi)(t_w) \circ \mathrm{Ad}(\psi)(\phi_0)$$

which is a composition of two translations $\mathrm{Ad}(\psi)(t_v) = t_{\psi_0(v)}$ and $\mathrm{Ad}(\psi)(t_w) = t_{\psi_0(w)}$ together with $\mathrm{Ad}(\psi)(\phi_0)$, whose linear part is $\phi_0' = \psi_0\phi_0\psi_0^{-1}$.
Since v is in the kernel of $\mathrm{Id} - \phi_0$, then

$$(\mathrm{Id}-\phi_0')(\psi_0(v)) = \psi_0(v)-\phi_0'(\psi_0(v)) = \psi_0(v)-\psi_0\phi_0\psi_0^{-1}\psi_0(v) = \psi_0(v-\phi_0(v)) = 0$$

from which $\psi_0(v) \in \ker(\mathrm{Id} - \phi_0')$.
Analogously, since $w \in \mathrm{Im}(\mathrm{Id} - \phi_0)$, we may write $w = (\mathrm{Id} - \phi_0)(u)$ for certain u and conclude that

$$\psi_0(w) = \psi_0(\mathrm{Id}-\phi_0)(u) = (\psi_0-\psi_0\phi_0)(u) = (\psi_0-\psi_0\phi_0\psi_0^{-1}\psi_0)(u) = (\mathrm{Id}-\psi_0\phi_0\psi_0^{-1})(\psi_0(u))$$

belongs to $\mathrm{Im}(\mathrm{Id} - \phi_0')$, just as the additional translation that may appear in $\mathrm{Ad}(\psi)(\phi_0)$ from (2.3).

Thus, by the classification of affine isometries in the lecture notes, one can check that the different types of isometries are preserved by conjugation since

$$\begin{array}{rcl}
\phi_0 = \mathrm{Id} & \Leftrightarrow & \phi_0' = \psi_0\phi_0\psi_0^{-1} = \mathrm{Id}, \\
\phi_0 = -\mathrm{Id} & \Leftrightarrow & \phi_0' = \psi_0\phi_0\psi_0^{-1} = -\mathrm{Id}, \\
v \neq 0 & \Leftrightarrow & \psi_0(v) \neq 0, \\
\ker(\phi_0 - \mathrm{Id}) = \{0\} & \Leftrightarrow & \ker(\phi_0' - \mathrm{Id}) = \ker(\psi_0(\phi_0 - \mathrm{Id})\psi_0^{-1}) = \{0\}, \\
\mathrm{Im}(\phi_0 - \mathrm{Id}) = \mathbb{R}^3 & \Leftrightarrow & \mathrm{Im}(\phi_0' - \mathrm{Id}) = \mathrm{Im}(\psi_0(\phi_0 - \mathrm{Id})\psi_0^{-1}) = \mathbb{R}^3, \\
& \text{and} & \\
\phi_0' & = & \psi_0\phi_0\psi_0^{-1}
\end{array}$$

and thus they are both the same kind of linear isometry (rotation, anti-rotation, symmetry with respect to a plane, etc).

2. The angle θ is computed up to sign by the formula

$$\cos(\theta) = \frac{1}{2}(\mathrm{tr}(\phi_0) - \det(\phi_0))$$

and both the trace and the determinant are preserved by conjugation. In the case of a rotation, the axis is the line of fixed points, so it must be $\psi(\ell)$ where ℓ is the axis of ϕ.

In the case of an anti-rotation, its axis is the line passing through the unique fixed point of ϕ' (which is the image by ψ of the unique fixed point of ϕ) with the direction of the eigenvector of eigenvalue -1 of ϕ_0' (which is the image by ψ_0 of the corresponding eigenvector of ϕ_0).

3. In general, P is a fixed point of φ if and only if $\psi(P)$ is a fixed point of $\varphi' = \psi \circ \varphi \circ \psi^{-1}$.

Exercice 10.

Exercice 11. 1. Determiner la matrice dans la base canonique de la rotation lineaire r d'angle $\pi/6$ et d'axe $\mathbb{R}(1,1,1)$.

2. Soit l'isometrie affine $r' = t_{(1,0,-1)} \circ r$. Quelle est la nature de r', ces eventuels points fixes et calculer $(r')^{2018}$.
3. Meme question pour $r'' = t_{(2,2,2)} \circ r$.

Preuve:

1. In a positively oriented orthonormal basis $B = \{v_1, v_2, v_3\}$ where $\mathbb{R}v_1 = \mathbb{R}(1,1,1)$ is must be the rotation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/6) & -\sin(\pi/6) \\ 0 & \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

To find such a basis one normalizes v_1 to $v_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, then one finds an orthogonal unitary vector like $v_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and completes the basis with the cross product $v_3 = v_1 \times v_2 = (-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$. Another way is with the Gram–Schmidt process.

Finally, one computes the matrix in the canonical basis using the base change matrix $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. The matrix under the canonical base is therefore given by

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1}{3} & \frac{1+\sqrt{3}}{3} \end{pmatrix}.$$

2. The vector $(1, 0, -1)$ is orthogonal to $(1, 1, 1) \in \ker(r - \mathrm{Id})$ and hence belongs to $\mathrm{Im}(r - \mathrm{Id})$. By the classification 6.2.1 (2) it is a rotation around the line of fixed points of angle $\pi/6$.

   The axis of rotation of r' is equal to $D_{r'} = -z + D_0$ (see Proposition 3.17), where D_0 denotes the axis of rotation r, and z is defined to be a vector such that $(1, 0, -1) = (r - \mathrm{Id})(z)$. For instance, one can take $z = (0, -1 - \sqrt{3}, 1)$. Therefore the axis of rotation of r' is

$$\mathrm{Fix}(r') = -z + \mathbb{R}(1, 1, 1).$$

   To calculate $(r')^{2018}$, since $2018 = 168 \times 12 + 2$, $(r')^{2018}$ gives 168 full rotations, followed by two rotations of $\pi/6$, which results in a rotation of $\pi/3$ around the axis described above.

3. This time $(2, 2, 2) \in \ker(r - \mathrm{Id})$. By the classification 6.2.1 (3), r'' is the composition of the affine rotation r around the axis $\mathbb{R}(1, 1, 1)$, followed by a translation with translation vector $(2, 2, 2)$, which is a vissage along $\mathbb{R}(1, 1, 1)$.

   Note that $t_{(2,2,2)} \circ r = r \circ t_{(2,2,2)}$ (see Theorem 3.12 (2)), then

$$(r'')^{2018} = (t_{(2,2,2)} \circ r)^{2018} = t_{(2,2,2)}^{2018} \circ r^{2018} = t_{(2,2,2)}^{2018} \circ r^{12 \cdot 168 + 2} = t_{(4036,4036,4036)} \circ r^2.$$