Neural Networks and Biological Modeling

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Question set 1

Exercise 1: Passive Membrane

The voltage across a passive membrane can be described by the equation

$$\tau \frac{du}{dt} = -(u - u_{\text{rest}}) + R\, I(t)\,. \tag{1}$$

1.1 Step current

Consider a current $I(t) = 0$ for $t < t_0$ and $I(t) = I_0$ for $t > t_0$. Calculate the voltage $u(t)$, given that the neuron is at rest at time t_0. (Hint: Instead of solving the differential equation explicitly, try to construct the response to the step current along the lines: What is the value of $u(t)$ for $t \leq t_0$? What is the asymptotic value of $u(t)$ for $t \gg t_0$? What is the functional form and time scale of the transition?)

1.2 Pulse current

Consider a current pulse

$$I(t) = \begin{cases} 0 & \text{for } t < t_0 \text{ and } t > t_0 + \Delta \\ q/\Delta & \text{for } t \geq t_0 \text{ and } t < t_0 + \Delta\,, \end{cases} \tag{2}$$

where Δ is a short time and q is the total electrical charge.

Consider first $\Delta = 0.1\tau$, and then $\Delta = 0.05\tau$, $\Delta = 0.025\tau$. Sketch the input current pulse and the voltage response. What happens in the limit $\Delta \to 0$? (Hint: Use $e^{-x} \approx 1 - x$ for $x \ll 1$.)

1.3 Delta function

The Dirac delta function can be defined by the limit of a short pulse:

$$\delta(t - t_0) = \lim_{\Delta \to 0} f_\Delta(t) \quad \text{where} \quad f_\Delta(t) = \begin{cases} 1/\Delta & \text{for } t_0 \leq t < t_0 + \Delta \\ 0 & \text{otherwise}\,. \end{cases} \tag{3}$$

Convince yourself that the integral $\int_{t_1}^{t_2} \delta(t - t_0)\, dt$ is equal to one if $t_1 \leq t_0 < t_2$ and vanishes otherwise.

Express $I(t)$ in Eq. 1 using the δ-function for the case that an extremely short current pulse arrives at time t^f. Pay attention to the units!

1.4 General solution

Assuming that before a given time t_0 the current is null and the membrane potential is at rest, derive the general solution to Eq. (1) for arbitrary $I(t)$.

Exercise 2: Integrate-and-fire model

Consider the model of Eq. (1) with a threshold at u = $\vartheta > u_{\text{rest}}$. If the membrane potential reaches the threshold, the neuron is said to fire and the membrane potential is reset to u_{rest}. The injected current is a step of magnitude I_0:

$$I(t) = \begin{cases} 0 & t \leq t_0 \\ I_0 & t > t_0 \end{cases}$$

2.1 What is the minimal current to reach the threshold, assuming $u(t = 0) = u_{\text{rest}}$?

2.2 At what time will the voltage first reach the threshold?

2.3 Calculate the firing frequency f as a function of I_0.

The function $g(I_0)$ which gives the firing frequency as a function of the constant applied current is called gain function.

Exercise 3: Integrate-and-fire models

The general form of an integrate-and-fire model is

$$\frac{du}{dt} = F(u) + \frac{RI(t)}{\tau} \tag{4}$$

where $F(u)$ is an appropriate function and $I(t)$ is the injected current. Three popular choices for the function F are the following (see Fig.1);

Leaky integrate-and-fire $F(u) = -\frac{u - u_{\text{rest}}}{\tau}$

Quadratic integrate-and-fire $F(u) = k\frac{(u - u_{\text{rest}})(u - u_{\text{th}})}{\tau}$

Exponential integrate-and-fire $F(u) = \frac{-(u - u_{rest}) + \Delta e^{\frac{u - u_{th}}{\Delta}}}{\tau}$.

3.1 Identify the resting potential u_{rest} and the spike threshold u_{th} in Fig. 1.

3.2 Consider three different values u_1, u_2 and u_3 for the voltage such that (i) u_1 is below u_{rest} (the resting potential), (ii) u_2 is between u_{rest} and u_{th} (the spike threshold), and (iii) u_3 is above u_{th} (see Fig. 1). For the three models described above, determine qualitatively the evolution of $u(t)$ when started at u_1, u_2, and u_3, assuming that the external input $I(t) \equiv 0$.

- For $u(t = 0) = u_1$, the voltage increases/decreases slowly/rapidly.
- For $u(t = 0) = u_2$, ...
- For $u(t = 0) = u_3$, ...

3.3 Why is u_{rest} called the resting potential? What is the role of u_{th}?

3.4 Consider the two voltage traces shown in Fig. 2(b) (top) in response to a step current (bottom). Using the graphs in Fig. 2(a), determine which of the two models was used to generate each trace.

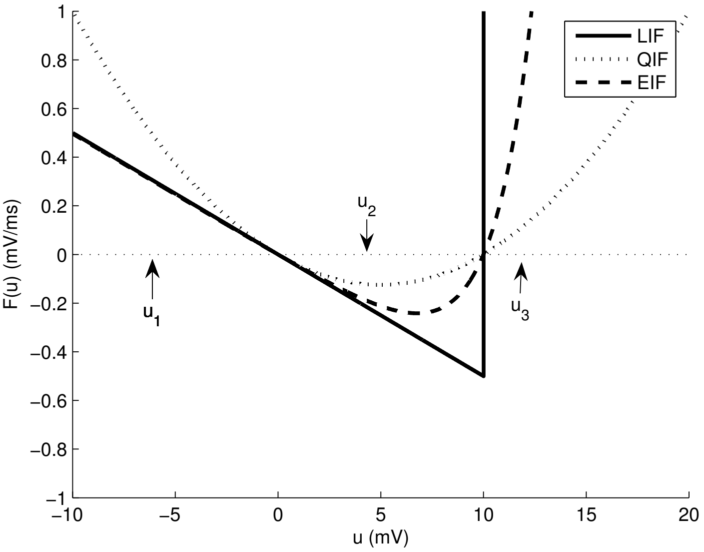
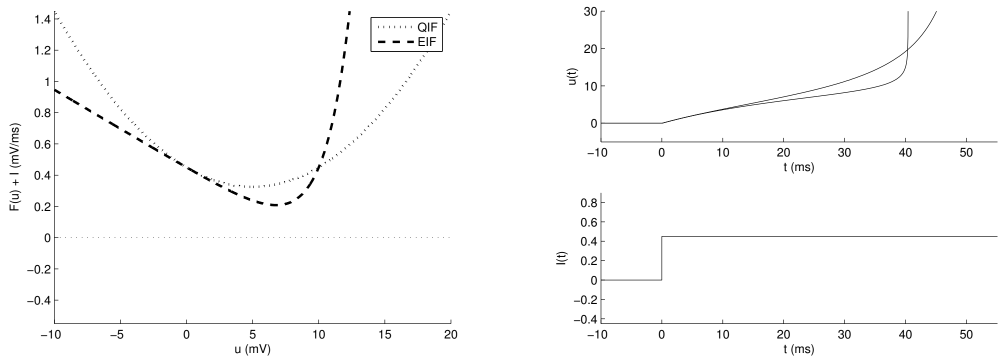


Figure 1: Sketch of the function $F(u)$ for three popular integrate-and-fire models.



(a)

(b)

Figure 2: Left: Right-hand side of Eq. 4 for the quadratic and exponential integrate-and-fire models if a constant input current $I(t) > 0$ is applied. Lower right: Trace of the injected current. Upper right: Voltage trace of the two models (EIF and QIF).