Neural Networks and Biological Modeling

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Correction Question Set 7

Exercise 1

1.1 The fixed point h_0 of the activity is defined by a loop of two closed equations: First, the mean firing rate $f = g(h_0)$ and second the population activity in the stationary state $A(t) = A_0 = g(h_0)$ (which is valid because the network is homogeneous and we have asynchronous firing).
For each neuron in the population we must have $h_0 = RI_0$ in the stationary state. For each neuron's current we have

$$
\begin{aligned}
I_i(t) &= I^{\text{ext}}(t) + \sum_j \sum_f w_{ij}\alpha(t - t_j^f) \\
I_i(t) &= I^{\text{ext}}(t) + \frac{J_0}{N} N \int \alpha(s)A(t-s)ds \\
I_i(t) &= I^{\text{ext}}(t) + J_0 A_0
\end{aligned}
$$

Above we used the fact that the network is in the stationary state, has large N and the fact that we have all-to-all connectivity with same weights.There are two ways to see the last step: We have
$\int \alpha(s)A(t-s)ds$
which is just the definition of the *filtered population activity* $\overline{A}(t)$. For large populations, the fluctuations go to 0 and we have $\overline{A}(t) = A_0(t)$.
A second interpretation is to replace $A(t-s)$ by the constant (stationary) population activity A_0 and pull that constant out of the integral.
$A_0 \int \alpha(s)ds$
The integral is assumed to be normalized to 1.

For I_0 we have

$$
\begin{aligned}
I_0 &= I^{\text{ext}}(t) + J_0 g(h_0) \\
g(h_0) &= \frac{I_0 - I^{\text{ext}}(t)}{J_0} \\
g(h_0) &= \frac{h_0 - RI^{\text{ext}}(t))}{RJ_0}
\end{aligned}
$$

The fixed point of the activity is therefore given by the intersection between the curve $f = g(h_0)$ and the straight line defined by the last equation.

1.2 For $h_1 = 1$ and $h_2 = 2$ we have

$$
f = g(h) = \begin{cases} 0 & , \quad h < 1 \\ h - 1 & , \quad 1 \le h \le 2 \\ 1 & , \quad 2 < h \end{cases} \tag{1}
$$

With $R = 1$ and $I^{\text{ext}}(t) = 0$, we have $g(h_0) = \frac{h_0}{J_0}$. J_0 controls the slope of the line.
With $J_0 = 1$: one fixed point, with $J_0 = 3$: three fixed points.

At $J_0 = 2$ we transition from one fixed point to three. $I^{\text{ext}}(t) \neq 0$ controls the bias of the line. Qualitatively, depending on it we may have 0, 1 or 3 fixed points. See next question for the more precise analytical solution.

1.3 In order to give analytical values for h_0 (hence for $f_0 = g(h_0)$) at the fixed point of the dynamics, we have to consider all possible cases.

1. if the slope of the line $\frac{1}{RJ_0}$ is greater than the slope of the transfer function $1/(h_2 - h_1)$ then there is only one fixed point. We have three cases:
   - $f = 0$ is a fixed point if $RI_{\text{ext}} < h_1$
   - $f = 1$ is a fixed point if $RI_{\text{ext}} > h_2 - RJ_0$
   - $f = \frac{h_1 - RI_{\text{ext}}}{RJ_0 - (h_2 - h_1)}$ in between the two previous cases
2. Otherwise if $\frac{1}{RJ_0} < 1/(h_2 - h_1)$, we have 1 or 3 fixed points (we don't do the calculation here).

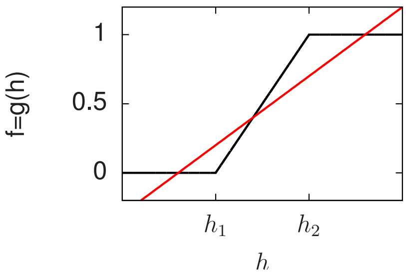

Figure 1: Graphical interpretation of a fixed point

Exercise 2

2.1 Following the steps of the previous exercise for the current, but now we consider the activity over a sub-population of K neurons. For a sub-population we have $A_k(t) = \frac{1}{K}\sum_k \sum_f \delta(t - t_k^f)$. Since the network is homogeneous, the sub-populations will also be homogeneous and we may assume that $A_k(t) \approx A_0$.

$$I_i(t) = \sum_k \sum_f w_{ik}\alpha(t - t_k^f)$$

$$I_i(t) = \sum_k \sum_f w_{ik} \int_0^\infty \alpha(s)\delta(t - t_k^f - s)ds$$

$$I_i(t) = \frac{w_0}{K} \int_0^\infty \alpha(s) \sum_k \sum_f \delta(t - t_k^f - s)ds$$

$$I_i(t) = \frac{w_0}{K} \int_0^\infty \alpha(s) K A_k(t - s)ds$$

$$I_i(t) = \frac{w_0}{K} K A_{k0} \int_0^\infty \alpha(s)ds$$

$$I_i(t) \approx w_0 A_0 \int_0^\infty \alpha(s)ds$$

To see the approximation from a more intuitive argument, assume a population of, say, 10'000 neurons. Each neuron receives input from K of them. We can say, each neuron draws K samples from the population. Because we have a homogenous network, all of these samples are statistically the same. We expect the sample mean to equal the population mean.

2.2 The current and it's fluctuations do not change because the weights do not scale with N. The fluctuations of the population activity $A(t)$ decrease.