Neural Networks and Biological Modeling

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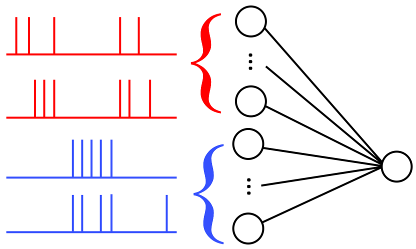
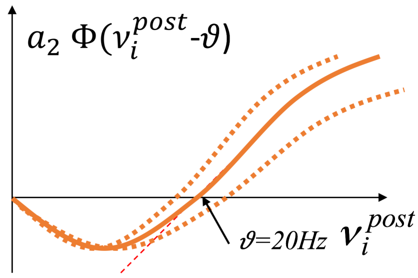
Question set 13

Exercise 1: Synaptic Plasticity: the BCM rule

A neuron receives 20 inputs that are organized in two groups of 10 inputs. The two groups fire in alternation: when group 1 is active, group 2 is silent; when group 2 is active, group 1 is silent. The input switches between the two groups every second (see figure 1(a)). All initial weights are $w_{ij} = 1$, but weights can change according to the BCM rule (eq. 1 with $\vartheta = 20Hz$). The firing rate of the postsynaptic neuron ν_i^{post} is given by eq. 2. The shape of Φ is shown in figure 1(b).



(a) One postsynaptic neuron receives input from 20 presynaptic neurons.



(b) weight-change as a function of ν_i^{post}. The solid line shows it for $\vartheta = 20Hz$.

Figure 1: Network and weight-change

$$\frac{d}{dt} w_{ij} = a_2^{corr} \Phi(\nu_i^{post} - \vartheta) \nu_j^{pre} \tag{1}$$

$$\nu_i^{post} = g(I_i) = \sum_j^N w_{ij} \nu_j^{pre} \tag{2}$$

a) Assume that group 1 fires at 3Hz, then group 2 at 1 Hz, then again group 1 etc. How do the weights of both groups evolve?

b) Assume that group 1 fires at 3Hz, then group 2 at 2.5 Hz, then again group 1 etc. How do the weights of both groups evolve?

c) The inputs are as in part b, but now you are free to choose theta. Suppose that the synapse can measure the time-average postsynaptic rate $\overline{\nu}$. What would you propose as model of ϑ so that the weight-pattern becomes non-trivial?

Exercise 2: Spike-time dependent plasticity by local variables

The goal of this exercise is to show that it is possible to account for the asymmetry in the STDP window using a simple microscopic model of synaptic plasticity.

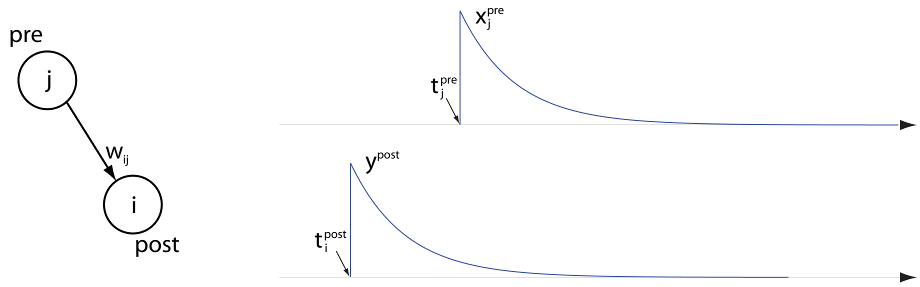


Figure 2: Memory traces of pre- and post-synaptic spike trains.

Suppose that the change in synaptic weight is controlled by the local concentration of two molecules x^{pre} and y^{post}. The substance x^{pre} acts as a memory trace of presynaptic spikes in the sense that each presynaptic spike triggers an increase in the concentration of x^{pre}:

$$\tau_+ \frac{d}{dt} x_j^{\mathrm{pre}} = -x_j^{\mathrm{pre}} + \delta(t - t_j^{\mathrm{pre}})\,. \tag{3}$$

Similarly, y^{post} is the trace left by the postsynaptic spike train,

$$\tau_- \frac{d}{dt} y^{\mathrm{post}} = -y^{\mathrm{post}} + \delta(t - t_i^{\mathrm{post}})\,. \tag{4}$$

Calculate the form of the learning window $\Delta w = f(\Delta t)$ – where $\Delta t = t_j^{\mathrm{pre}} - t_i^{\mathrm{post}}$ assuming that the synaptic weights are updated according to the rule

$$\frac{d}{dt} w_{ij} = a_+ x_j^{\mathrm{pre}} \delta(t - t_i^{\mathrm{post}}) - a_- y^{\mathrm{post}} \delta(t - t_j^{\mathrm{pre}})\,. \tag{5}$$

The constants a_+ and a_- are both positive.

Hint: Calculate the weight change for a pair of pre/post spikes. Consider the two cases $\Delta t > 0$ and $\Delta t < 0$.

Exercise 3: From spike-time dependent plasticity to rate models

Suppose that we have pair-based plasticity with an STDP window $W(t_i^f - t_j^{f'})$. The window decays exponentially and the slowest time scale of the decay is τ_-. Every presynaptic spike interacts with every postsynaptic spike as long as the timing is close enough to fall within the above time window.

3.1 Assume presynaptic spike trains generated by a homogeneous Poisson process with rate ν_j. Assume postsynaptic spike trains generated by another, independent, Poisson process with constant rate ν_i. How much is the expected weight change Δw_{ij} in a time T, if $T >> \tau_-$?

Hint: Write the weight change as an integral over spike trains. Link the expectation over spike trains to the firing rate.

3.2 Assume presynaptic spike trains generated by a homogeneous Poisson process with rate ν_j. Assume

postsynaptic spike trains generated by another Poisson process with rate:
$\nu_i(t) = \sum_k w_{ik} \sum_f \epsilon(t - t_k^f) = \sum_k w_{ik} \int_0^\infty \epsilon(s) S_k(t-s) ds$.
How much is the expected weight change $\Delta w_{ij}/T$ in a time T, if $T >> \tau_-$?

Hints:
(i) Exploit the autocorrelation of the Poisson process.
(ii) The output spikes are generated with rate ν_i, but this rate depends on the input.
(ii) Treat the input from synapse j explicitly. Note that the output spike train depends on the input spikes: If a spike has arrived at time t_j^f the postsynaptic rate is higher than 'on average'.

Exercise 4: Hopfield networks and Hebbian learning (TODO at home)

Here we explore how we may obtain a Hopfield network with M stored prototypes through Hebbian plasticity instead of fixing the weights explicitly.

This is achieved by presenting the patterns to a fully connected network and apply a plasticity rule:

$$\frac{d}{dt} w_{ij} = a_2^{\text{corr}} (\nu_i^{\text{post}}(t) - \vartheta)(\nu_j^{\text{pre}}(t) - \vartheta)\,, \tag{6}$$

where a_2 and ϑ are parameters of the plasticity model; $\nu_i^{\text{post}}(t)$ and $\nu_j^{\text{pre}}(t)$ are the activities of neurons i and j at time t.

We present a pattern μ to the network in the following way: Each pixel j of pattern μ, $p_j^\mu \in \{-1, +1\}$, stimulates exactly one neuron j in the network. That neuron's firing rate ν_j depends on the pattern: $\nu_j = 0\,Hz$ if $p_j^\mu = -1$; $\nu_j = 20\,Hz$ if $p_j^\mu = +1$.

During that presentation, the network learns the pattern by adjusting its weights according to the plasticity rule given in equation 6. We assume initial weights $w_{ij} = 0$. For this exercise, we use a constant threshold $\vartheta = 10\,Hz$.

4.1 We now have the network learn M patterns. Each one is presented once for 0.5 seconds. Show that, for an appropriate choice of a_2, the final weights are given by

$$w_{ij} = \sum_\mu p_i^\mu p_j^\mu\,. \tag{7}$$

Hint: Begin by calculating the weight change induced by presenting a single pattern for $0.5s$.

4.2 How does this learning rule map to the general formulation

$$\frac{d}{dt} w_{ij} = a_0 + a_1^{\text{pre}} \nu_j^{\text{pre}} + a_1^{\text{post}} \nu_i^{\text{post}} + a_2^{\text{corr}} \nu_j^{\text{pre}} \nu_i^{\text{post}} + \ldots ? \tag{8}$$

4.3 Would you describe this learning procedure as reinforcement or unsupervised learning?