Neural Networks and Biological Modeling

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Correction Question Set 9

Exercise 1 : Nullclines

1.1 For the nullclines of h_1, $\frac{d}{dt}h_1(t)$, we have

$$0 = -h_1(t) + h^{ext} + (w_{ee} - \alpha)g(h_1(t)) - \alpha g(h_2(t)) \quad (1)$$

i.e. numerically

$$0 = -h_1(t) + 0.8 + 0.5g(h_1(t)) - g(h_2(t)).$$

We can also solve for $g(h_2)$ along the h_1 nullcline:

$$\alpha g(h_2) = -h_1(t) + h^{ext} + (w_{ee} - \alpha)g(h_1(t))$$

The table is

h_1	$g(h_2)$	h_2
0	0.85	0.9
0.2	0.7	0.7
0.8	0.4	0.4
1	0.25	0.25

The steps for the h_2 nullcline are the same. The two nullclines are plotted in figure 1.

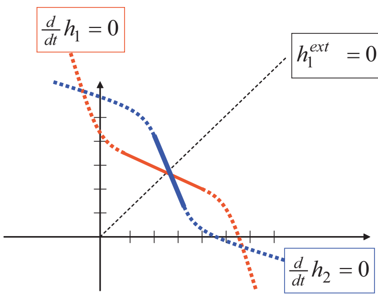

Figure 1.

1.2 & **1.3**
The arrows of the phase plane and the three trajectories can be seen in figure 2. As we can see from the direction of the arrows and as we will show in the next exercise, the middle fixed point is a saddle point. The point (0,0) belongs to the stable manifold of the saddle point, so the trajectory that starts from there moves along this manifold, is attracted towards the saddle point and stops when it reaches it. The other two trajectories are pushed away from the saddle point and depending on their starting point go to one of the other two (stable) fixed points.

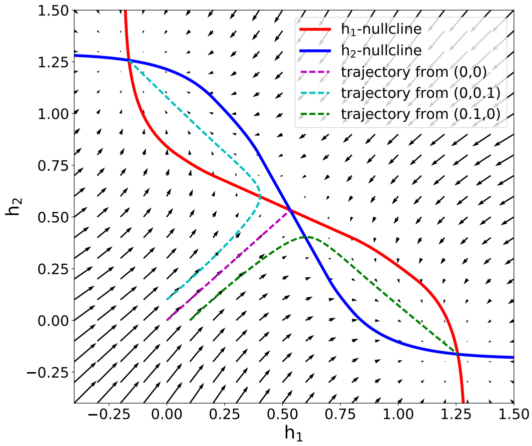

Figure 2.

Exercise 2 : Stability of the homogeneous solution

2.1 Rewriting the system of equation with the numerical values ($g(h) = h$, $w_{ee} = 1.5$, $\alpha = 1$):

$$\frac{d}{dt}h_1 = b - 0.5h_1 - h_2 \tag{2}$$

$$\frac{d}{dt}h_2 = b - 0.5h_2 - h_1. \tag{3}$$

The nullclines are given by:

$$h_2 = b - 0.5h_1$$

$$h_1 = b - 0.5h_2.$$

Since the system of equation is symmetric, there is a fixed point that follows the symmetry at $h^* \equiv h_1 \equiv h_2$, solving yields:

$$h^* = \frac{2}{3}b.$$

One could analyze the stability of the system given in 2. With a view to exercise 2.3, we want to keep track of the parameters w_{ee} and α, and start from the original system given in the exercise sheet. The stability of that nonlinear system is analyzed by linearizing it around the fixed-point of interest and studying the eigenvalues of the Jacobian matrix:

$$\begin{pmatrix} \frac{d}{dt}h_1 \\ \frac{d}{dt}h_2 \end{pmatrix} \approx \begin{pmatrix} \frac{\partial}{\partial h_1}(\frac{dh_1}{dt}) & \frac{\partial}{\partial h_2}(\frac{dh_1}{dt}) \\ \frac{\partial}{\partial h_1}(\frac{dh_2}{dt}) & \frac{\partial}{\partial h_2}(\frac{dh_2}{dt}) \end{pmatrix}_{|(h^*,h^*)} \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \end{pmatrix}$$

Plugging in the expressions given on the exercise sheet we get the following Jacobian:

$$J = \begin{pmatrix} -1 + (w_{ee} - \alpha)\frac{\partial}{\partial h_1}g(h_1) & -\alpha\frac{\partial}{\partial h_2}g(h_2) \\ -\alpha\frac{\partial}{\partial h_1}g(h_1) & -1 + (w_{ee} - \alpha)\frac{\partial}{\partial h_2}g(h_2) \end{pmatrix}_{|(h^*,h^*)} \tag{4}$$

We have $(w_{ee} - \alpha) = (1.5 - 1) = 0.5$, $g(h) = h$, $\frac{d}{dh}g(h^*) = 1$. Therefore

$$J = \begin{pmatrix} -0.5 & -1 \\ -1 & -0.5 \end{pmatrix}$$

For a 2x2 matrix, it is convenient to express the eigenvalues in terms of its determinant and trace:

$$\lambda_{\pm} = \frac{\mathrm{Tr}\, J \pm \sqrt{(\mathrm{Tr}\, J)^2 - 4\mathrm{Det}\, J}}{2}. \tag{5}$$

With

$$\mathrm{Tr}\, J = -2 + 2(w_{ee} - \alpha)g'(h^*)$$

$$\mathrm{Det}\, J = 1 - 2(w_{ee} - \alpha)g'(h^*) + (w_{ee}^2 - 2w_{ee}\alpha)g'(h^*)^2$$

we find $\lambda_- = -1.5$ and $\lambda_+ = 0.5$. We look at the sign of the two real eigenvalues to determine the stability: (pos/pos) → unstable, (neg/neg) → stable, (pos/neg) → saddle point. Hence, for the given parameters, the fixed point (h^*, h^*) is unstable, and it is a saddle point.

2.2 We have $h_1 = h_2 = h^*$ and

$$0 = -h^* + b + (w_{ee} - \alpha)g(h^*) - \alpha g(h^*)$$

From which we find the following implicit equation for the fixed points in the general case:

$$h^* = b + (w_{ee} - 2\alpha)g(h^*)$$

2.3 We express the Jacobian found above (eq. 4) using β and the given parameters. Simplifying gives:

$$J = \begin{pmatrix} -1+\beta & -\beta \\ -\beta & -1+\beta \end{pmatrix}$$

With $Det(J) = 1 - 2\beta$ and $Tr(J) = -2 + 2\beta$. Using eq. 5 we find

$$\lambda_{\pm} = -1 + \beta \pm \beta.$$

2.4 In the case where the fixed point is in the region where $g'(h^*) = 0$ we have $\lambda_{\pm} = -1$. This corresponds to a stable fixed point. In the case where $g'(h^*) = 1$ then $\lambda_- = -1$ and $\lambda_+ = \frac{1}{2}$. The fixed point is unstable. In fact the point is stable if and only if $\beta < \frac{1}{2}$.

2.5 With $w_{ee} = 3/2$ and $\alpha = 3/4$ we have $h^* = b$. Then $\beta = \frac{3}{4}g'(h^*) = \frac{3}{4}g'(b)$. At $b = 0.8$ we have $g'(b) = 1$ and so $\beta = 3/4$. This is larger than 1/2 so according to the previous question the fixed point is a saddle point. As we decrease b from 0.8, $g'(h)$ decreases and eventually becomes 0. Then $\beta = 0 < 1/2$, so the fixed point becomes stable. Thus, for strong symmetric input ($b > 0.8$) the monkey is forced to make a decision, and it depends on the initial conditions and the noise in the input whether the system ends up left or right (as we saw in figure 2). But for weak symmetric input there is a stable fixed point and the monkey may not respond at all.