# Neural Networks and Biological Modeling

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## Question set 5

### Exercise 1

Consider a Hopfield network composed of 9 neurons. Each neuron has connections to all other neurons.

**1.1** How many connections are there in total? Choose the appropriate weights for the prototype pattern given in figure 1.

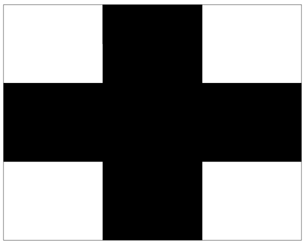

Figure 1: Prototype pattern. Black corresponds to S = +1.

Now keeping the learned weights fixed, present a pattern $S_i(t = 0)$ and let it evolve according to:

$$S_i(t + 1) = sign\left(\sum_j w_{ij} S_j(t)\right) \tag{1}$$

Suppose the initial state is again the swiss cross above but with one bit (neuron) flipped. Will the dynamics correct it?

**1.2** Suppose that $N$ bits are flipped. Will the dynamics correct them?

## Exercise 2: Associative memory

Consider a Hopfield network with a continuous state variable $S_i(t) \in \mathbb{R}$. Assume that the network has stored 4 patterns

$$\begin{aligned} p^1 &= \{p_1^1, \cdots, p_N^1\} \\ &\cdot \\ &\cdot \\ p^4 &= \{p_1^4, \cdots, p_N^4\} \end{aligned} \tag{2}$$

that are orthogonal, i.e., $\frac{1}{N}\sum_{i=1}^{N} p_i^\mu p_i^\nu = \delta^{\mu\nu}$, where $\delta^{\mu\nu}$ is the Kronecker symbol

$$\delta^{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

You present the network with an activity pattern that has overlap[1] with $p^3$ only (no overlap with other memories). The activity dynamics is given by

$$S_i(t+1) = g\left(\sum_j w_{ij} S_j(t)\right) \tag{3}$$

**2.1** Calculate the change of the overlap with pattern 3 in one time step, i.e. calculate $m^3(t+1)$ as a function of $m^3(t)$. Moreover, $g(\cdot)$ is an odd function: $g(-x) = -g(x)$

*Hint:* Follow the derivations shown in class (and in the book Neuronal Dynamics, chapter 17.2): Use the definitions of the overlap $m^3(t)$ and the weights $w_{ij}$ to express $S_i(t+1)$ (eq. 3) as a function of the overlap. Then, using $S_i(t+1)$ compute the overlap $m^3(t+1)$. Keep in mind that the state of each neuron always takes one of two values: $p_i \in \{-1, 1\}$.

**2.2** Use this to discuss the evolution of the overlap over several time steps

- when g is the sign-function
- when g is an odd and monotonically increasing function mapping the real line onto $[-1; 1]$. As an example, consider $g(x) = \tanh(\beta x)$ with some real, positive parameter $\beta$. Think about the effect of changing $\beta$ (sometimes called 'inverse temperature') and discuss the cases $\beta < 1$, $\beta > 1$ and $\beta \to \infty$.

## Exercise 3: Probability of error in the Hopfield model

**3.1** Consider a Hopfield network of N neurons (N = 10'000) storing P **random** prototypes $p^\mu$ and the following dynamics:

$$S_i(t+1) = sign\left(\sum_j w_{ij} S_j(t)\right) \tag{4}$$

Given the initial activation set to pattern 1, i.e. $S_i(t=0) = p_i^1$, show that

$$S_i(t=1) = p_i^1 sign(1 + \sum_{\mu \neq 1}^{P} \sum_{j}^{N} \frac{1}{N}\, p_i^1\, p_i^\mu\, p_j^1\, p_j^\mu). \tag{5}$$

[1] by "having overlap with prototype $\mu$" we mean with "having non-zero scalar product with $p^\mu$"

*Hint:* Start with the dynamics equation 4. Use the definition of the weights $w_{ij}$ to express the update in terms of the patterns.
*Hint:* You can always multiply a term with 1. In particular, with $1 = p_i^1 p_i^1$

**3.2** In equation 5, formulate the condition for which $S_i$ will change its state.
That is, $S_i(t = 1) \neq S_i(t = 0)$.

**3.3** Using the analogy for the sum as a random walk, show that the term $\sum_{\mu \neq 1}^{P} \sum_{j}^{N} \frac{1}{N} \; p_i^1 \; p_i^\mu \; p_j^1 \; p_j^\mu$ can be approximated by a Gaussian random variable, $N(0, (P-1)/N)$.
*Hint:* Specify mean and variance of the distribution of the random variable $X = p_i^1 \; p_i^\mu \; p_j^1 \; p_j^\mu$. Then use the central limit theorem to approximate the sum by a Gaussian.

**3.4** Show that the probability that a given neuron $i$ will flip ($S_i(t = 1) \neq S_i(t = 0)$) is given by

$$P_{\text{error}} = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{N}{2(P-1)}}\right)\right] \tag{6}$$

where erf is the error function, defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x'^2} dx' \,. \tag{7}$$

**3.5** How many random patterns can you store, if you accept on average at most 1 bit to be wrong? Consider $\text{erf}(2.6) = 0.9998$.

**3.6** In many real application, patterns to be stored are not totally random and have substantial overlap. Rewrite the retrieval equation 5 as a function of overlap terms, $m^{\mu\nu} = \frac{1}{N} \sum_i p_i^\mu p_i^\nu$.

**3.7** Assume that the overlap between different patterns is 0.1 for all pairs. How many patterns can you store now, allowing on average only one wrong bit?

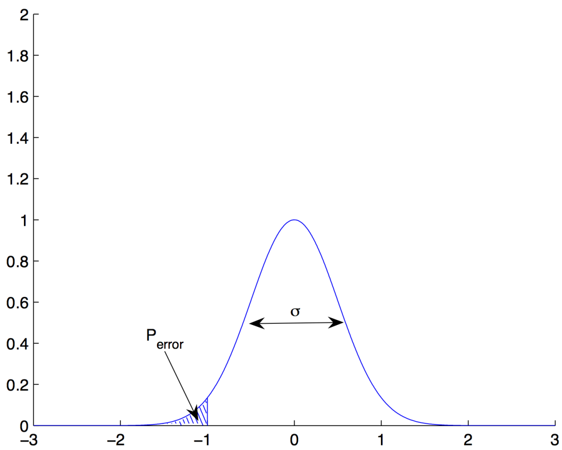


Figure 2: Error probability: $P(x \leq -1)$