Biological Modeling of Neural Networks

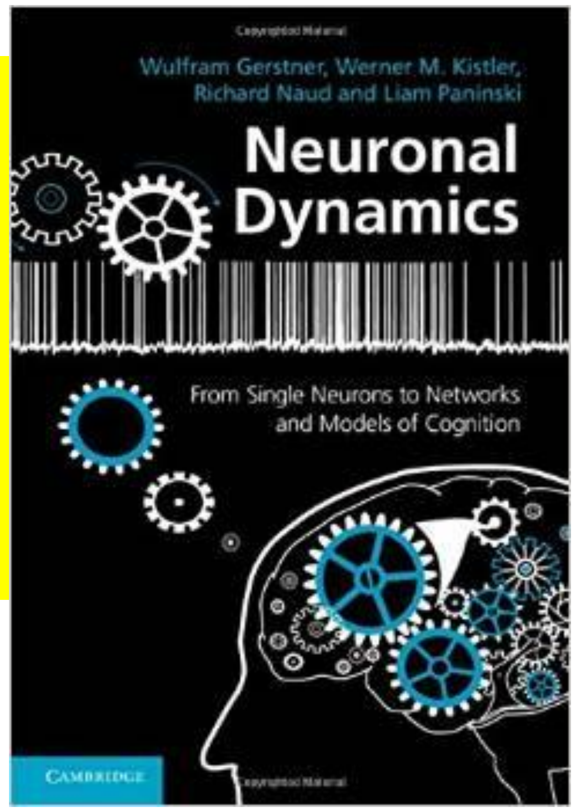
ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Week 3 – Reducing detail:
Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

Reading for week 3:
NEURONAL DYNAMICS
- Ch. 4.1- 4.3



Cambridge Univ. Press

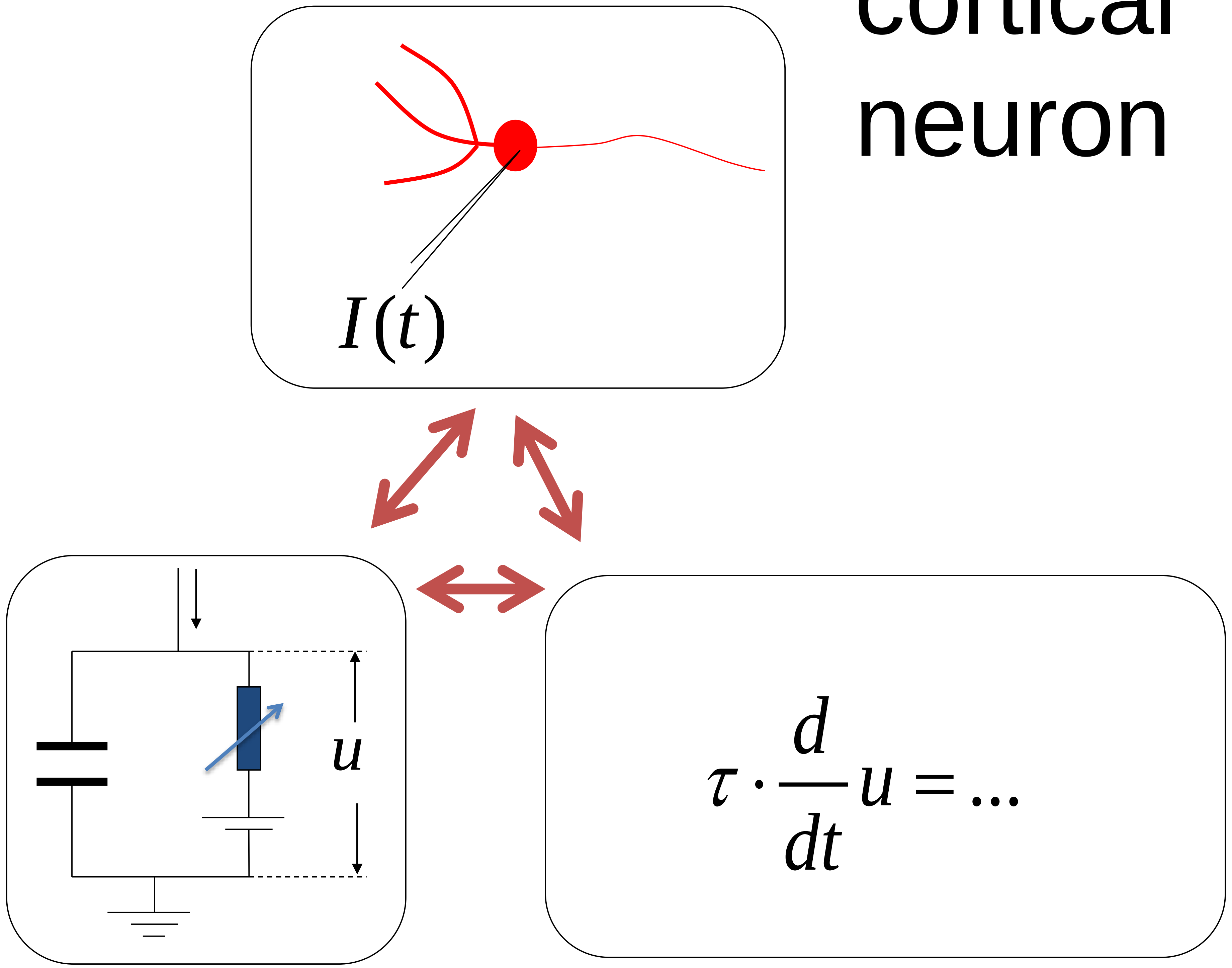
3.1 **From Hodgkin-Huxley to 2D**
- Overview: From 4 to 2 dimensions
- MathDetour 1: Exploiting similarities
- MathDetour 2: Separation of time scales

3.2 **Phase Plane Analysis**
- Role of nullclines

3.3 **Analysis of a 2D Neuron Model**
- constant input vs pulse input
- MathDetour 3: Stability of fixed points

3.1. Review of week 2 :Hodgkin-Huxley Model

cortical neuron



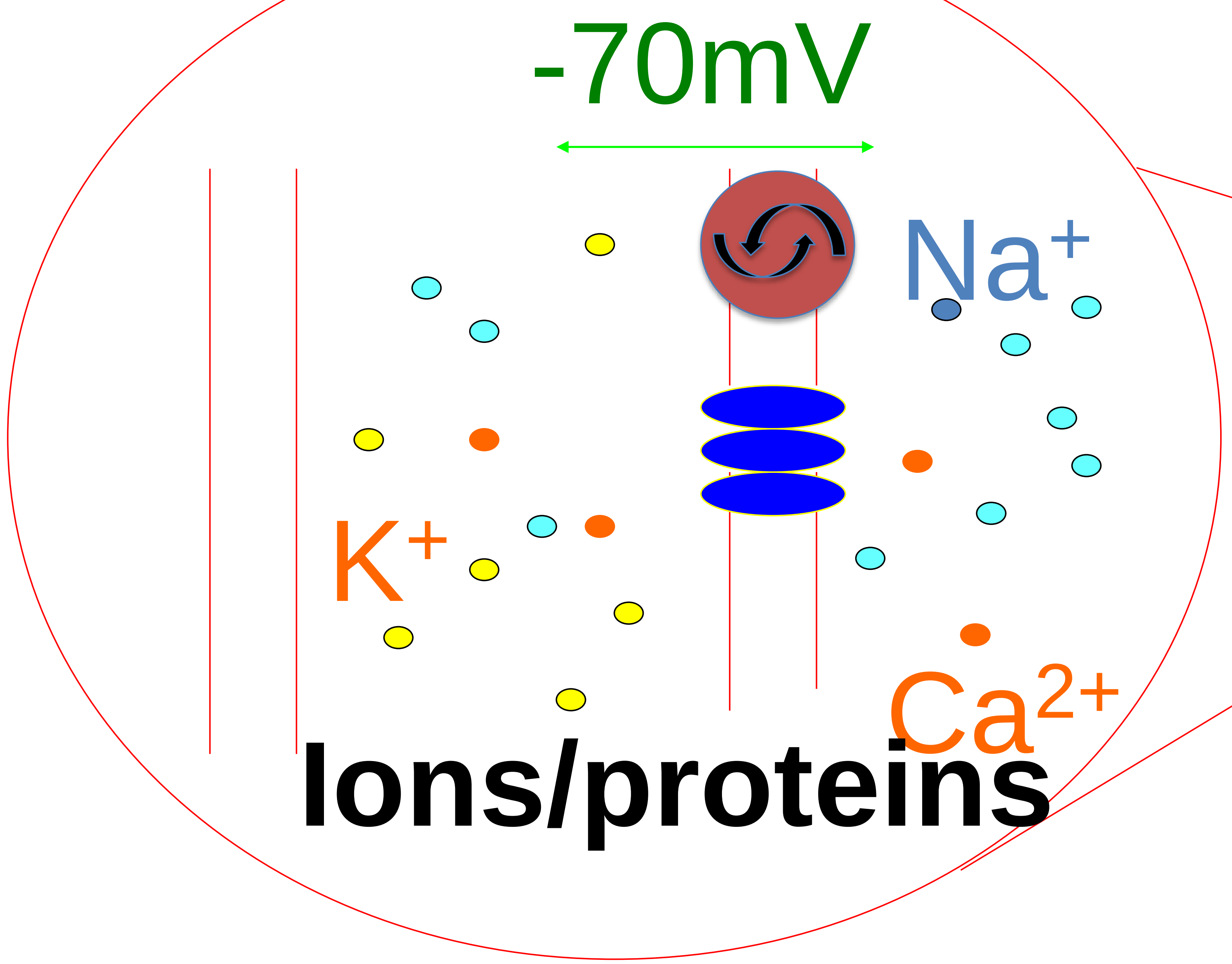
→Hodgkin-Huxley model
→Compartmental models

3.1 Review of week 2 : Hodgkin-Huxley Model

Week 2:

Cell membrane contains

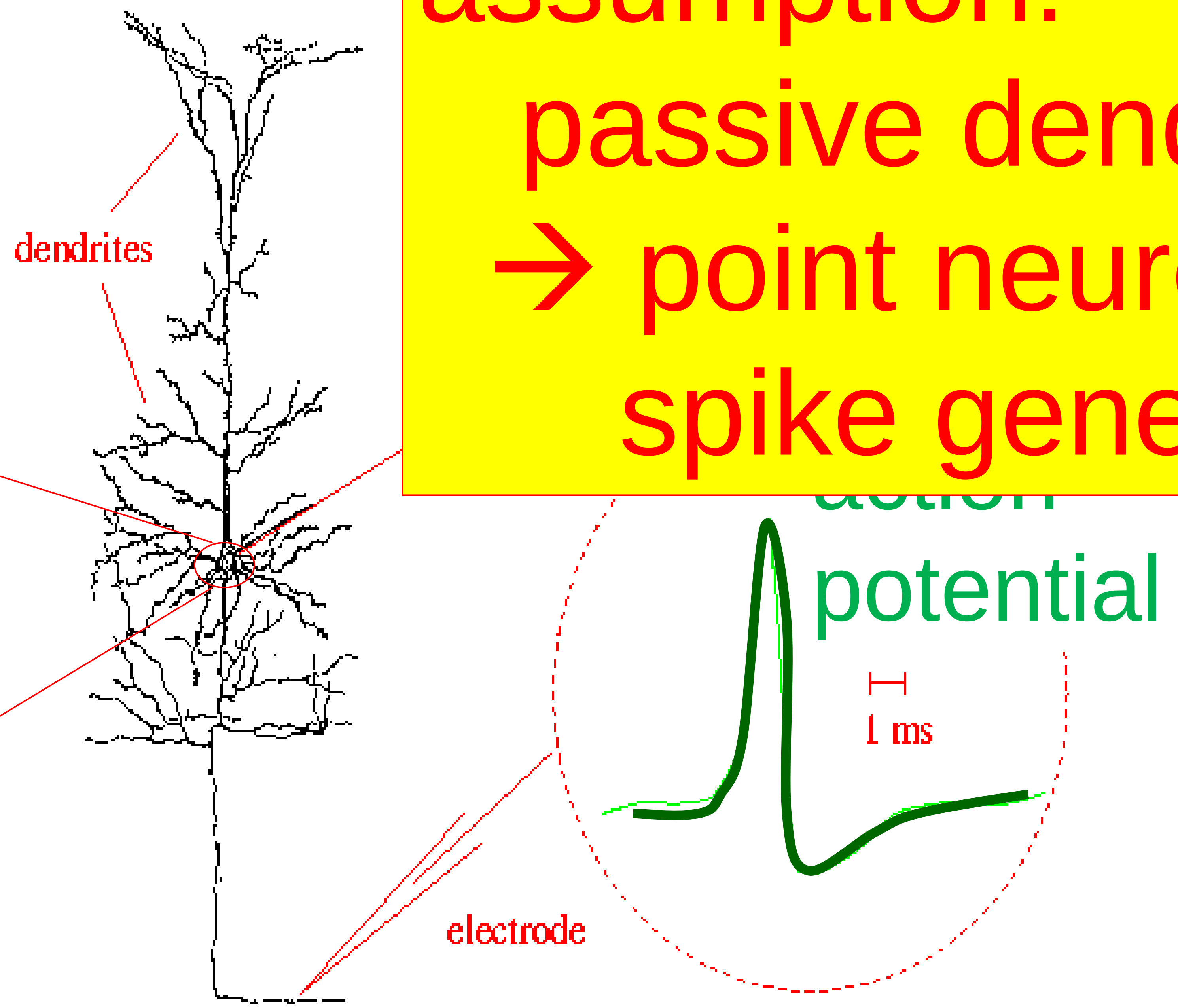
- ion channels
- ion pumpsa



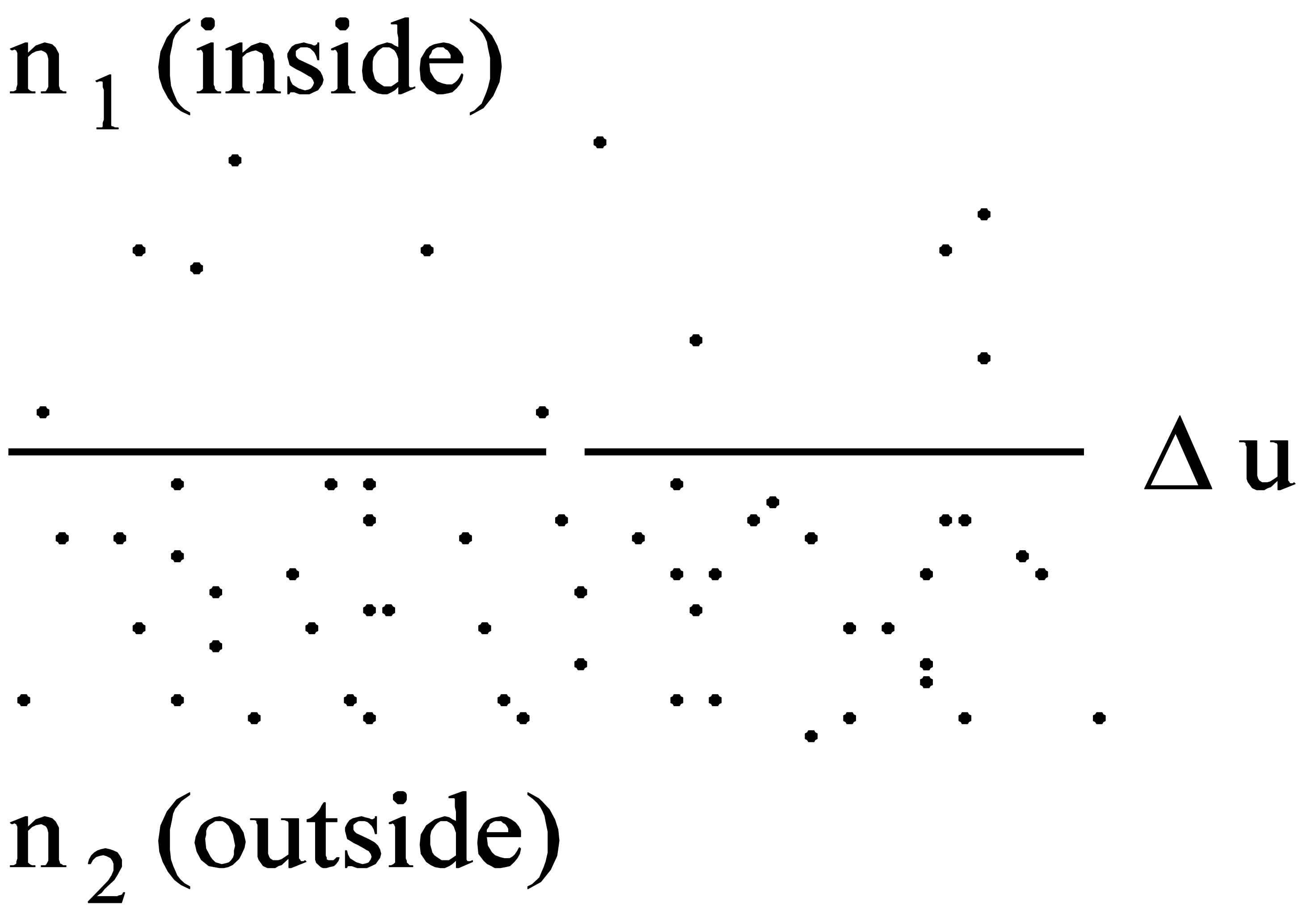
Dendrites (week x:video):

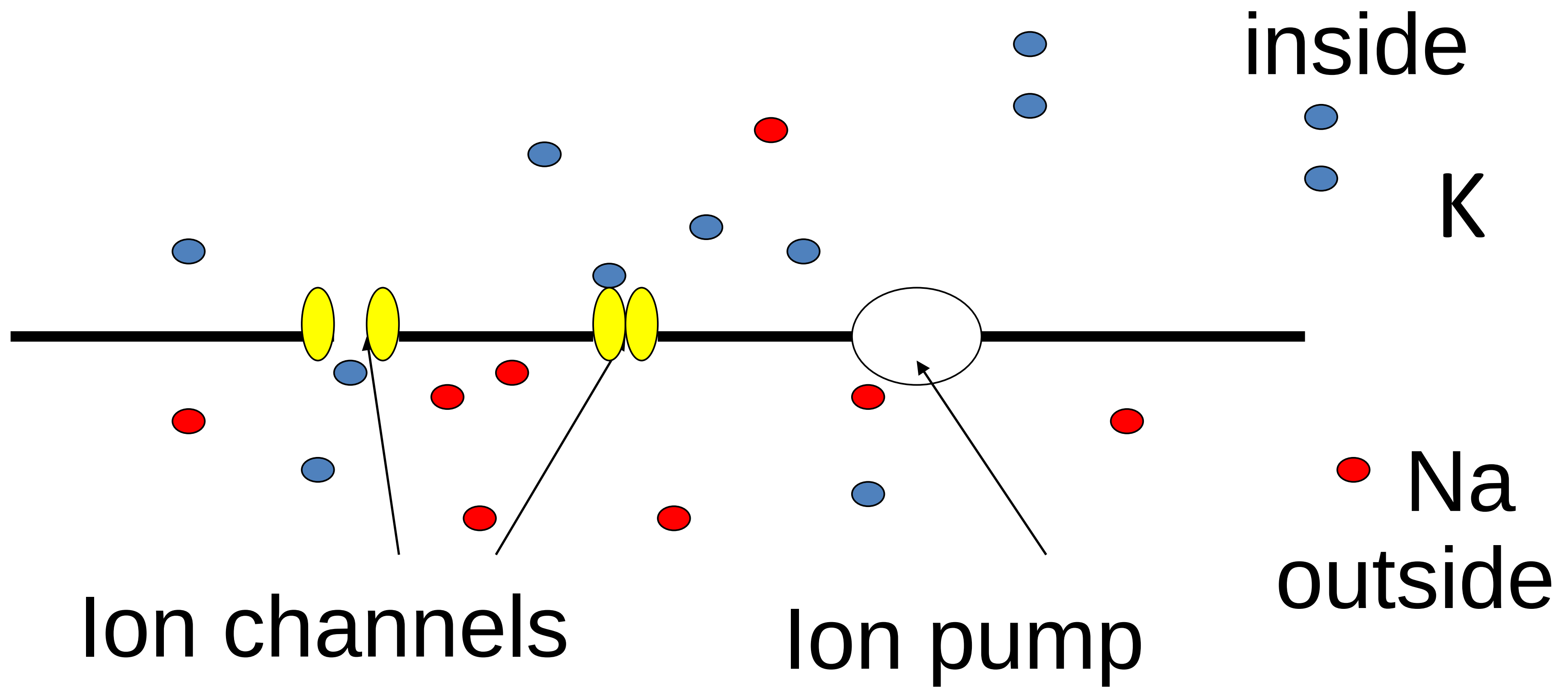
Active processes?

assumption:
passive dendrite
→ point neuron
spike generation



3.1. Review of week 2 :Hodgkin-Huxley Model





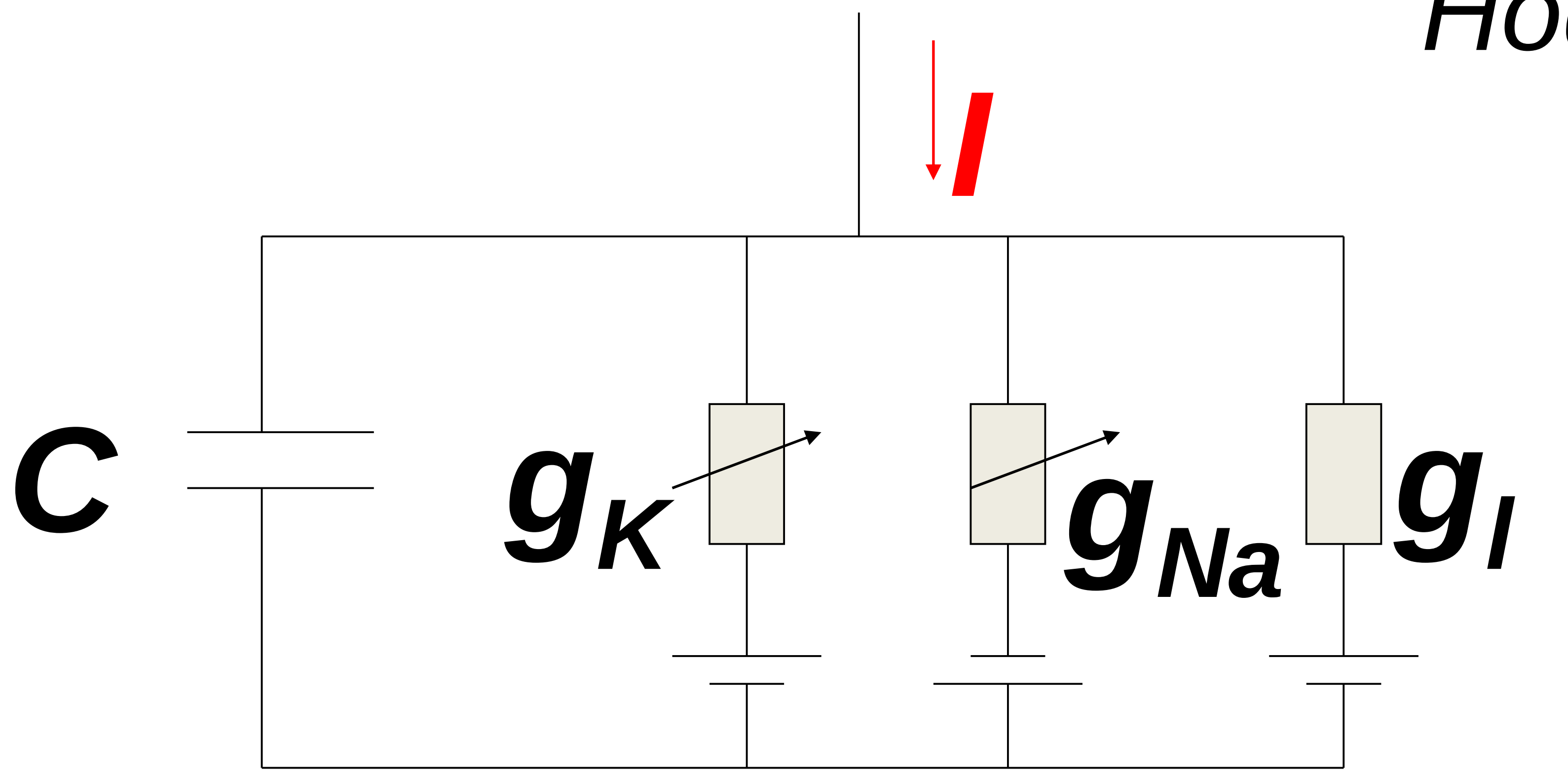
$$\Delta u = u_1 - u_2 = \frac{-kT}{q} \ln \frac{n(u_1)}{n(u_2)}$$

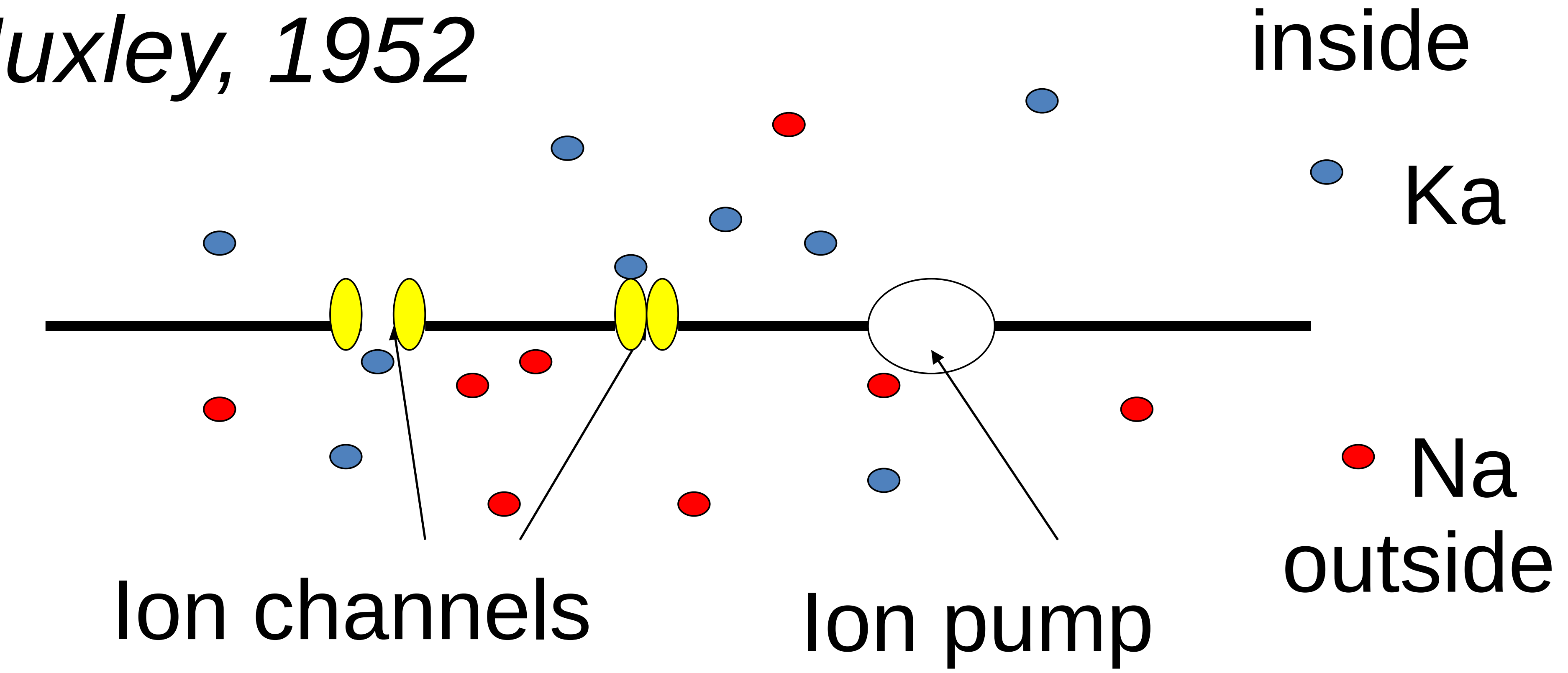
Reversal potential

ion pumps →concentration difference ⇔ voltage difference

3.1. Review of week 2: Hodgkin-Huxley Model

Hodgkin and Huxley, 1952



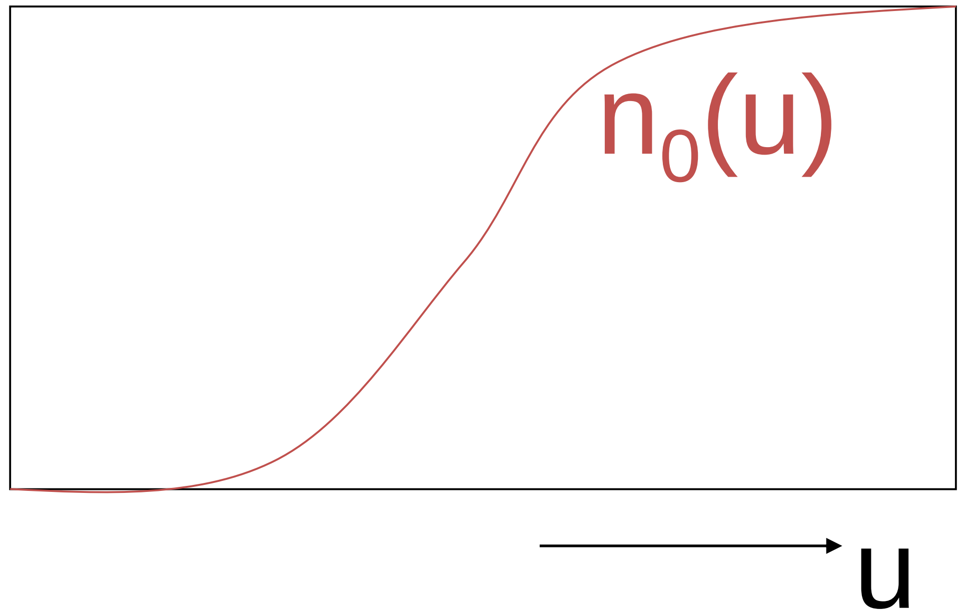


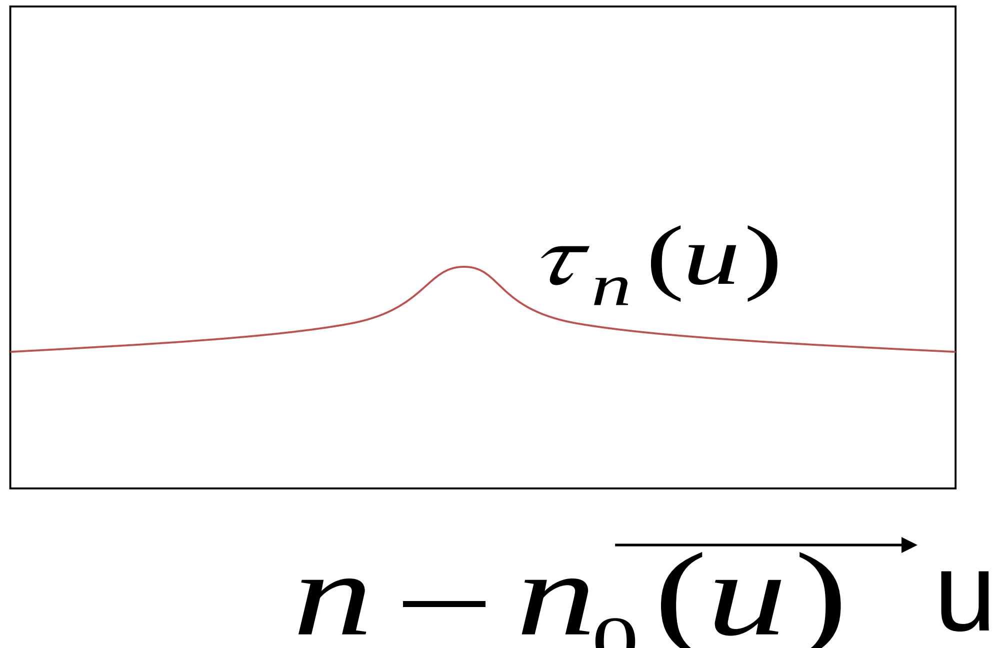
$$C\frac{du}{dt} = \overbrace{-g_{Na}m^3h(u-E_{Na})}^{I_{Na}}\overbrace{-g_K n^4(u-E_K)}^{I_K}\overbrace{-g_l(u-E_l)}^{I_{leak}}+\overset{\text{stimulus}}{I(t)}$$

4 equations
= 4D system

$$\frac{dm}{dt} = -\frac{m-m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$





$$\frac{dn}{dt} = -\frac{n-n_0(u)}{\tau_n(u)}$$

Week 3 – 3.1. Overview and aims

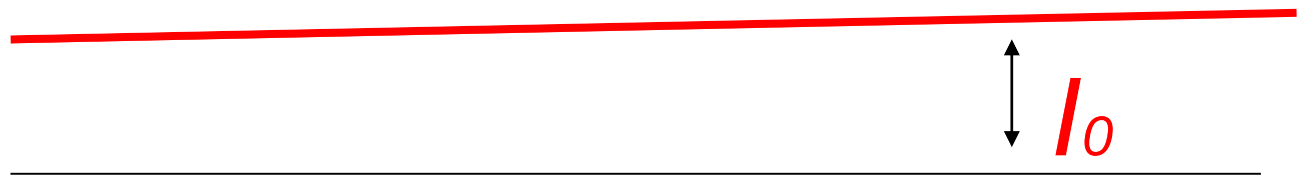
Can we understand the dynamics of the HH model?

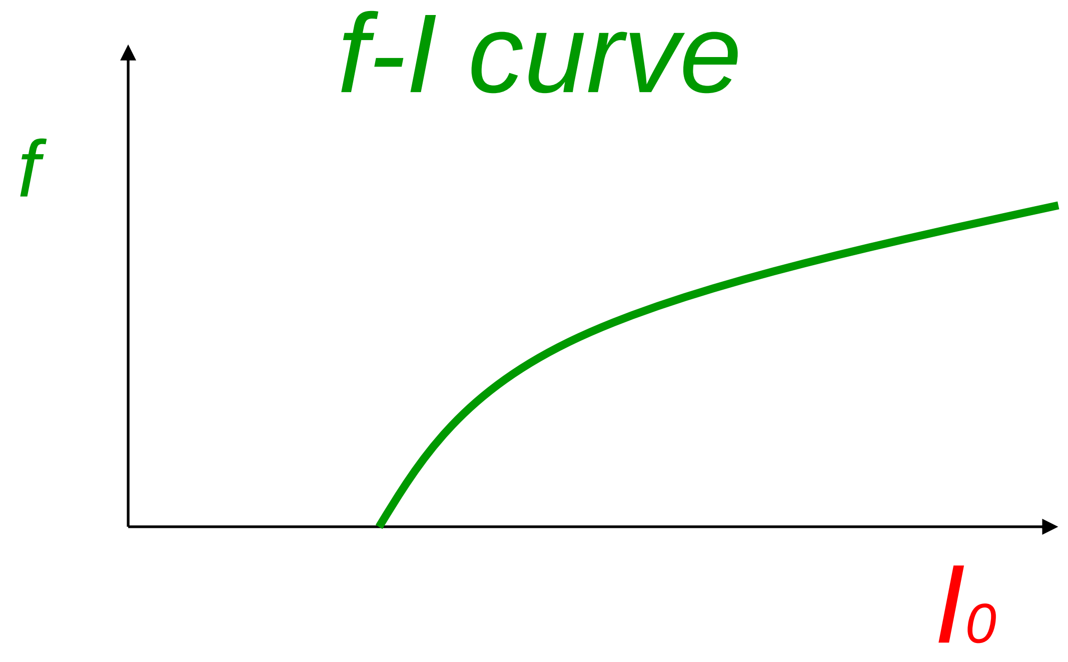
- mathematical principle of Action Potential generation?
- constant input current vs pulse input?
- Types of neuron model (type I and II)? (next week)
- threshold behavior? (next week)

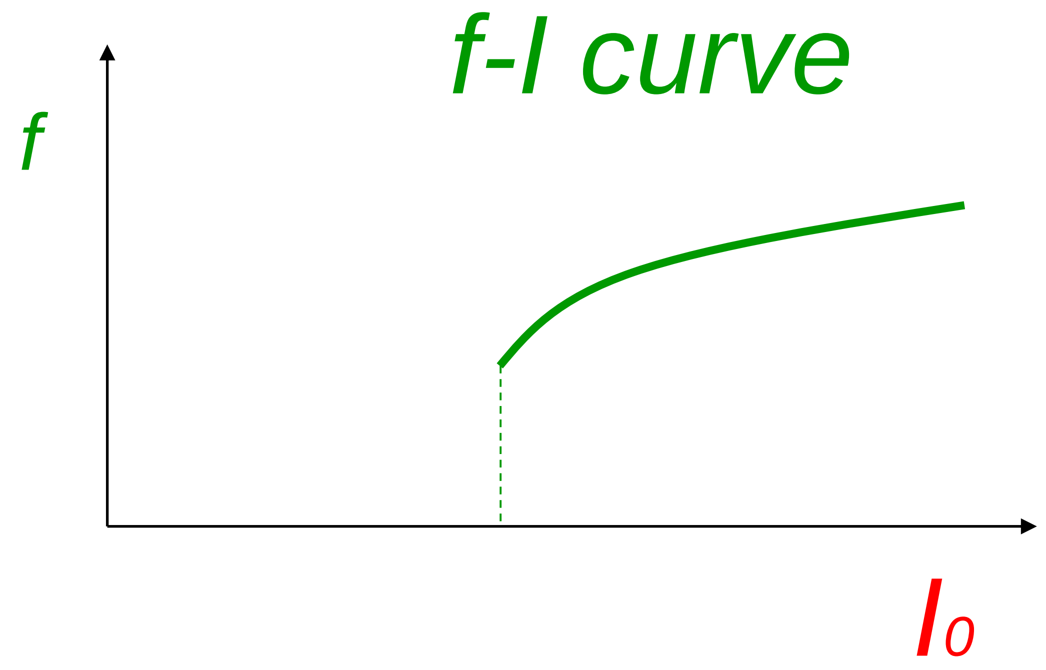
→ Reduce from 4 to 2 equations

Type I and type II models

ramp input/
constant input







Week 3 – 3.1. Overview and aims

Can we understand the dynamics of the HH model?

→ Reduce from 4 to 2 equations

Week 3 – Quiz 3.1.

A - A biophysical point neuron model
with 3 ion channels,
each with activation and inactivation,
has a total number of equations equal to

[ ] 3 or
[ ] 4 or
[ ] 6 or
[ ] 7 or
[ ] 8 or more

Toward a two-dimensional neuron model

-**Reduction of Hodgkin-Huxley to 2 dimension**

-step 1: separation of time scales

-step 2: exploit similarities/correlations

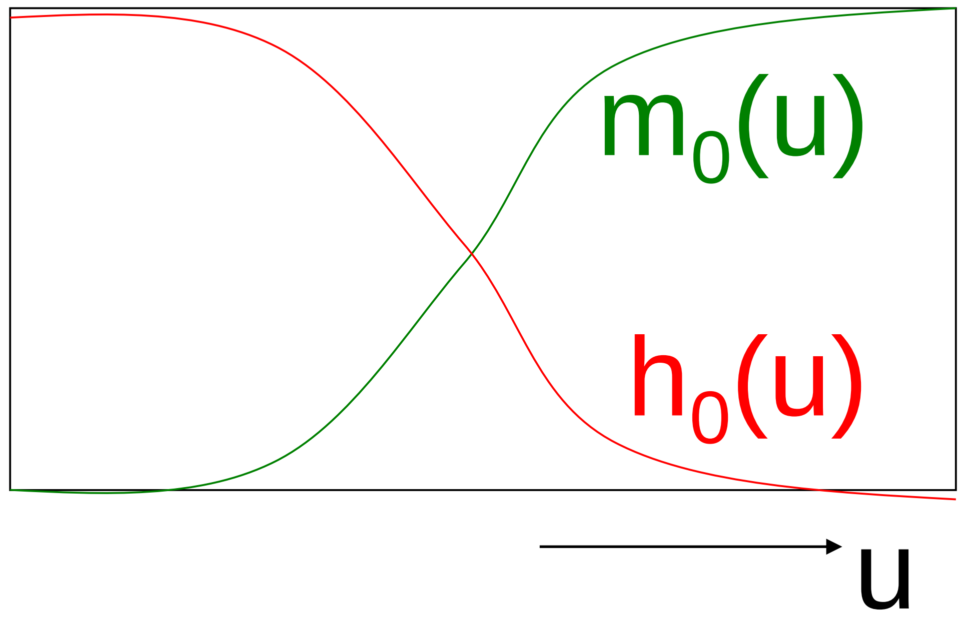
3.1. Reduction of Hodgkin-Huxley model

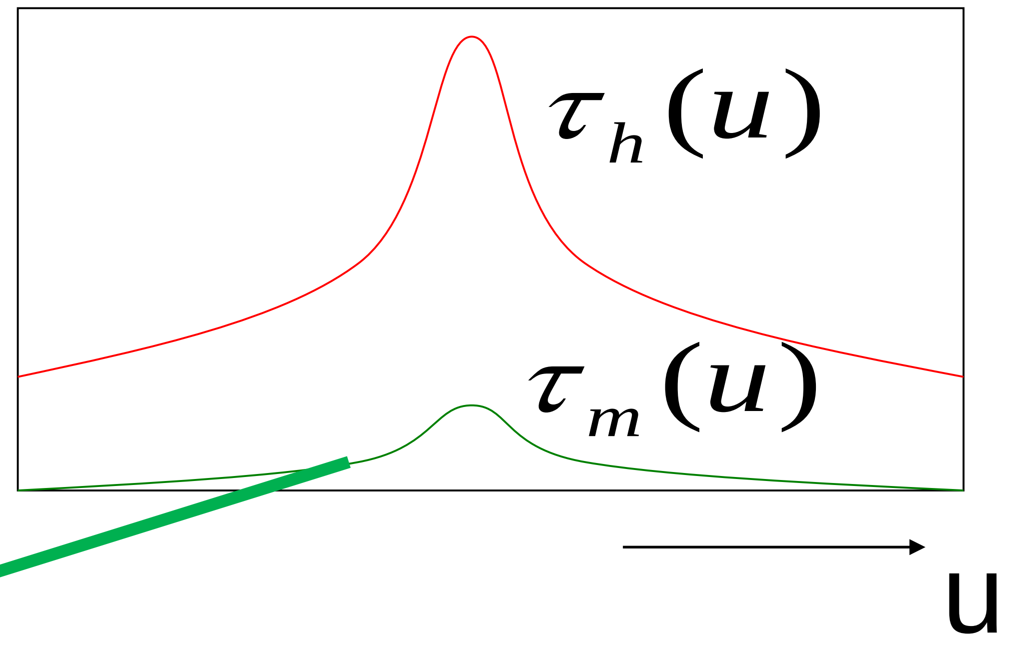
$$C\frac{du}{dt} = -\overbrace{g_{Na} m^3 h (u - E_{Na})}^{I_{Na}} - \overbrace{g_K n^4 (u - E_K)}^{I_K} - \overbrace{g_l (u - E_l)}^{I_{leak}} + \overset{\text{stimulus}}{I(t)}$$

$$\frac{dm}{dt} = -\frac{m - m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n - n_0(u)}{\tau_n(u)}$$





Details later!

1) dynamics of *m* are fast $\longrightarrow$ $m(t) = m_0(u(t))$

Reduction of dimensionality: Separation of time scales

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

Exercise 1 (week 3)
at beginning of lecture 2 !

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + h(y)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$\tau_1 \ll \tau_2 \rightarrow x=h(y)$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(h(y))$$

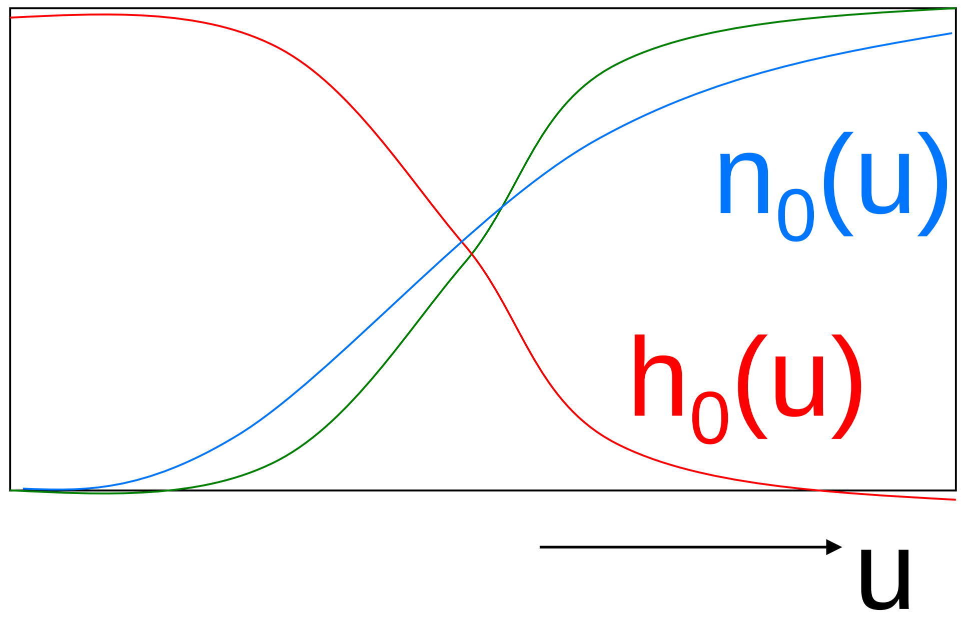
3.1. Reduction of Hodgkin-Huxley model

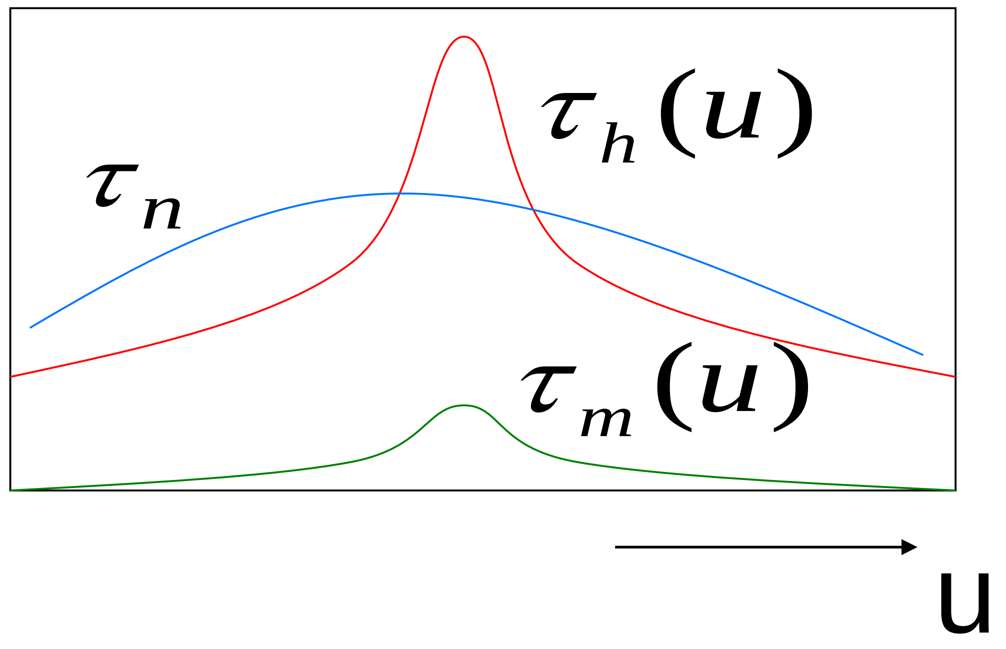
$$C\frac{du}{dt} = -\overbrace{g_{Na}m^3h(u-E_{Na})}^{I_{Na}} - \overbrace{g_K n^4(u-E_K)}^{I_K} - \overbrace{g_l(u-E_l)}^{I_{leak}} + \overset{\substack{\text{stimulus}\\ \downarrow}}{I(t)}$$

$$\frac{dm}{dt} = -\frac{m-m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n-n_0(u)}{\tau_n(u)}$$





1) dynamics of *m* are fast $\longrightarrow$ $m(t) = m_0(u(t))$

2) dynamics of *h* and *n* are similar

Reduction of Hodgkin-Huxley Model to 2 Dimension

-step 1:
separation of time scales

-step 2:
exploit similarities/correlations

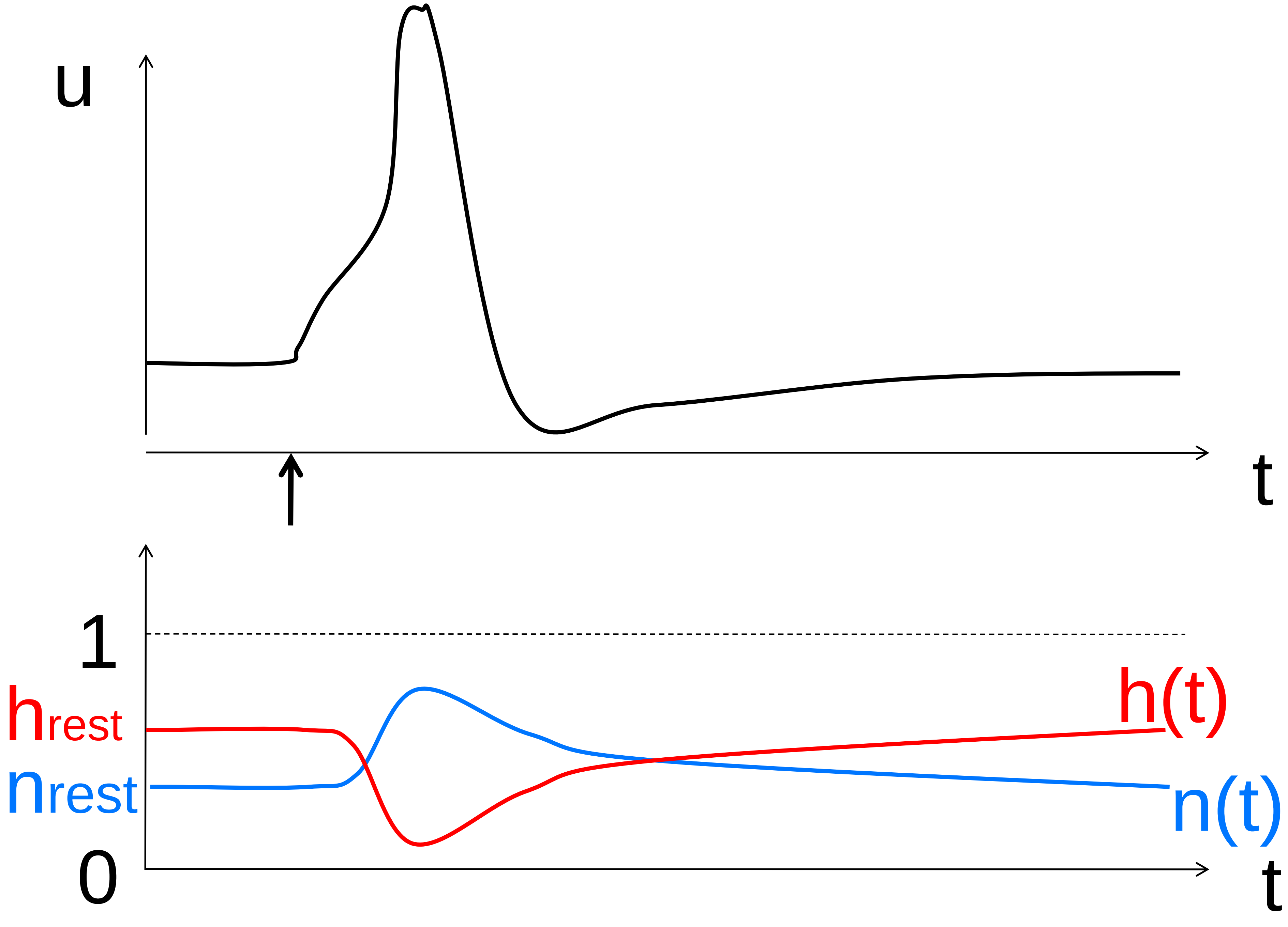
Now !

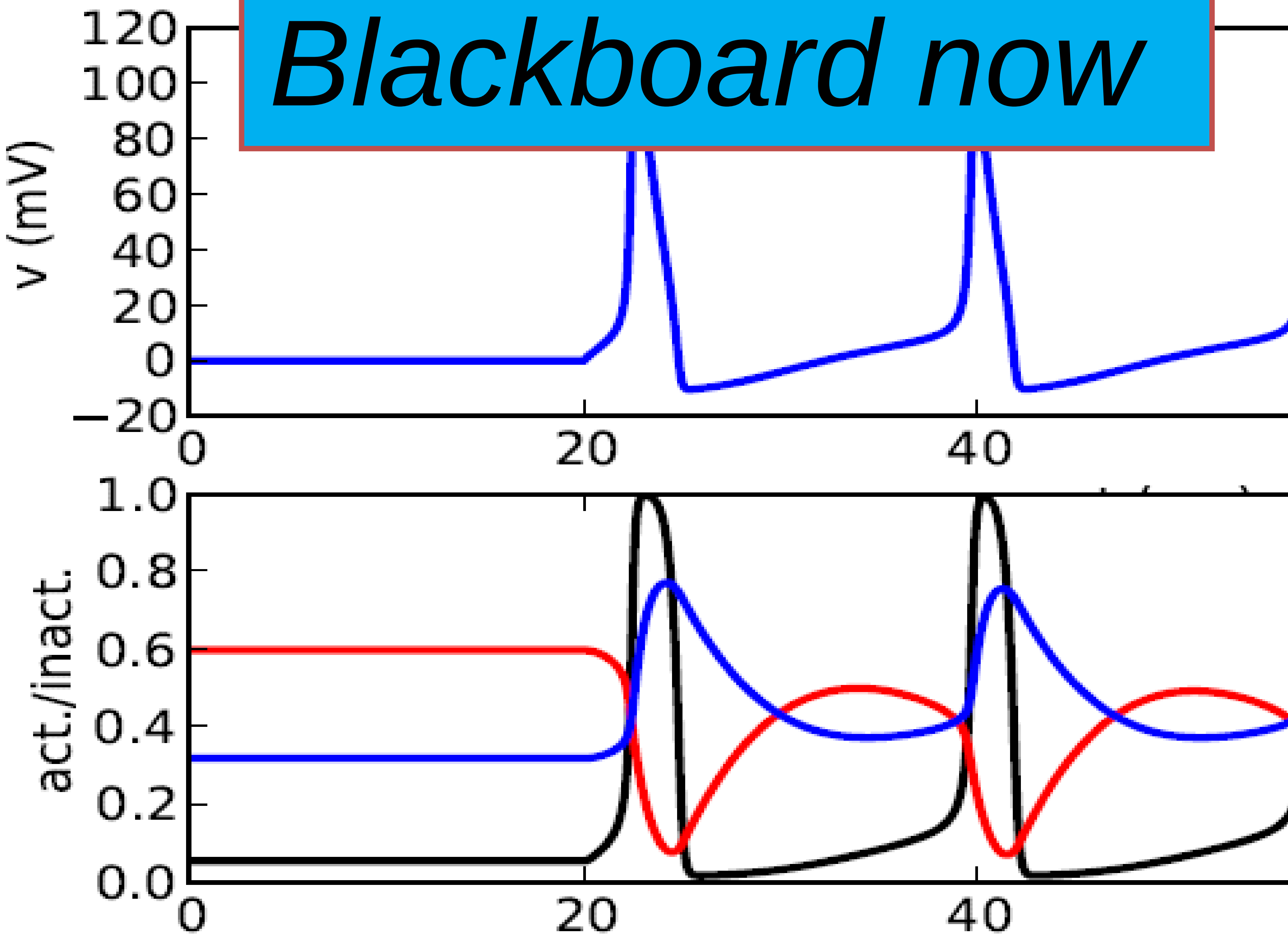
3.1. Reduction of Hodgkin-Huxley model

stimulus

$$C\frac{du}{dt} = -g_{Na}m^3h(u-E_{Na}) - g_K n^4(u-E_K) - g_l(u-E_l) + I(t)$$

2) dynamics of *h* and *n* are similar $\longrightarrow$ $1-h(t) = a\,n(t)$





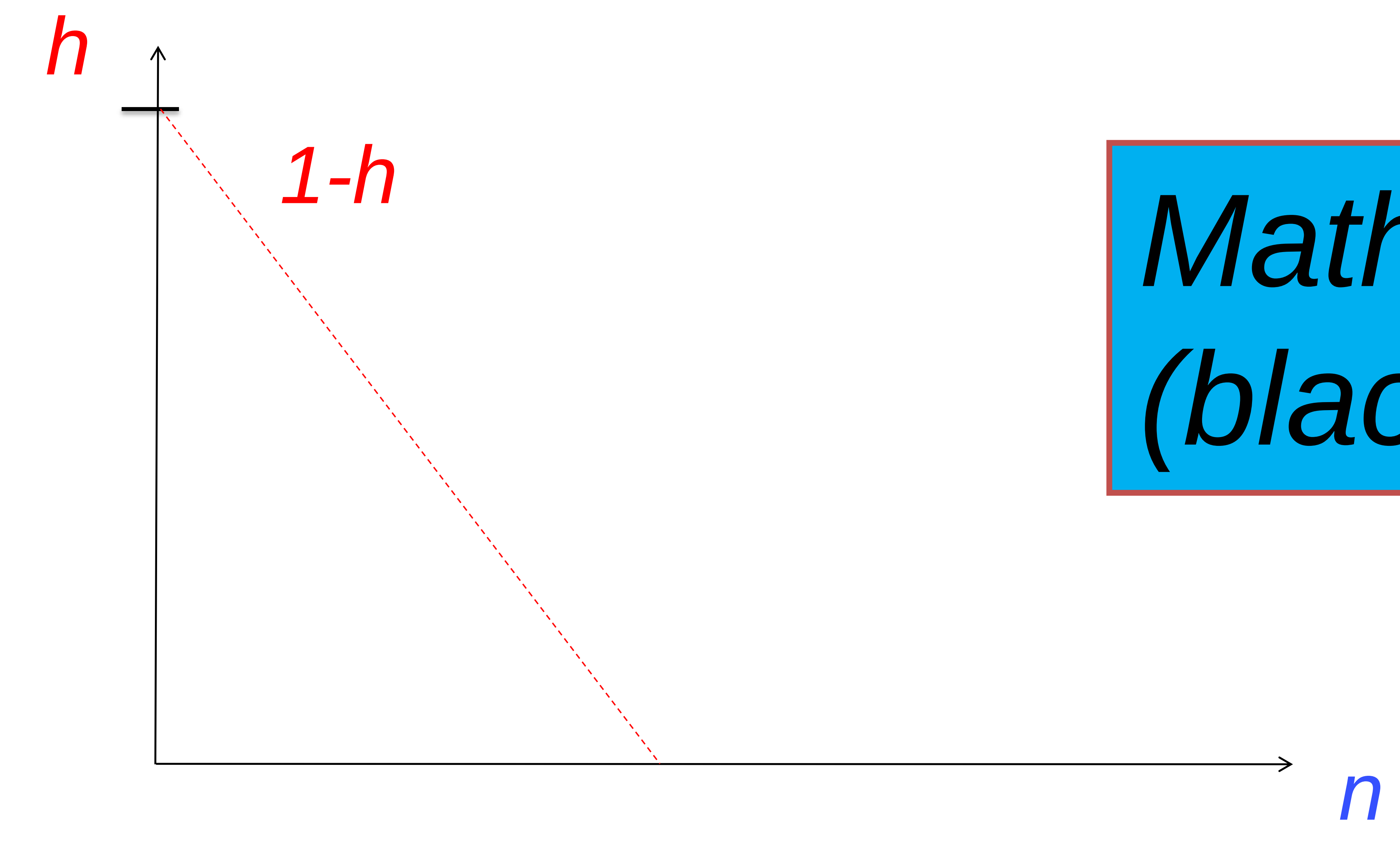
Blackboard now

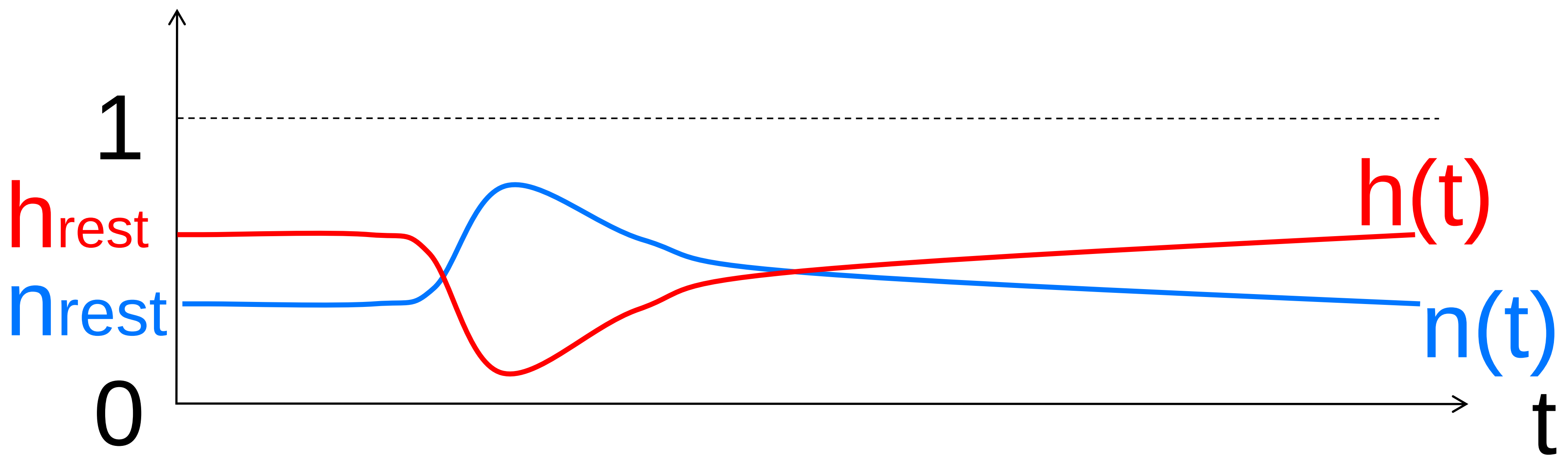
3.1 Detour 1. Exploit similarities/correlations

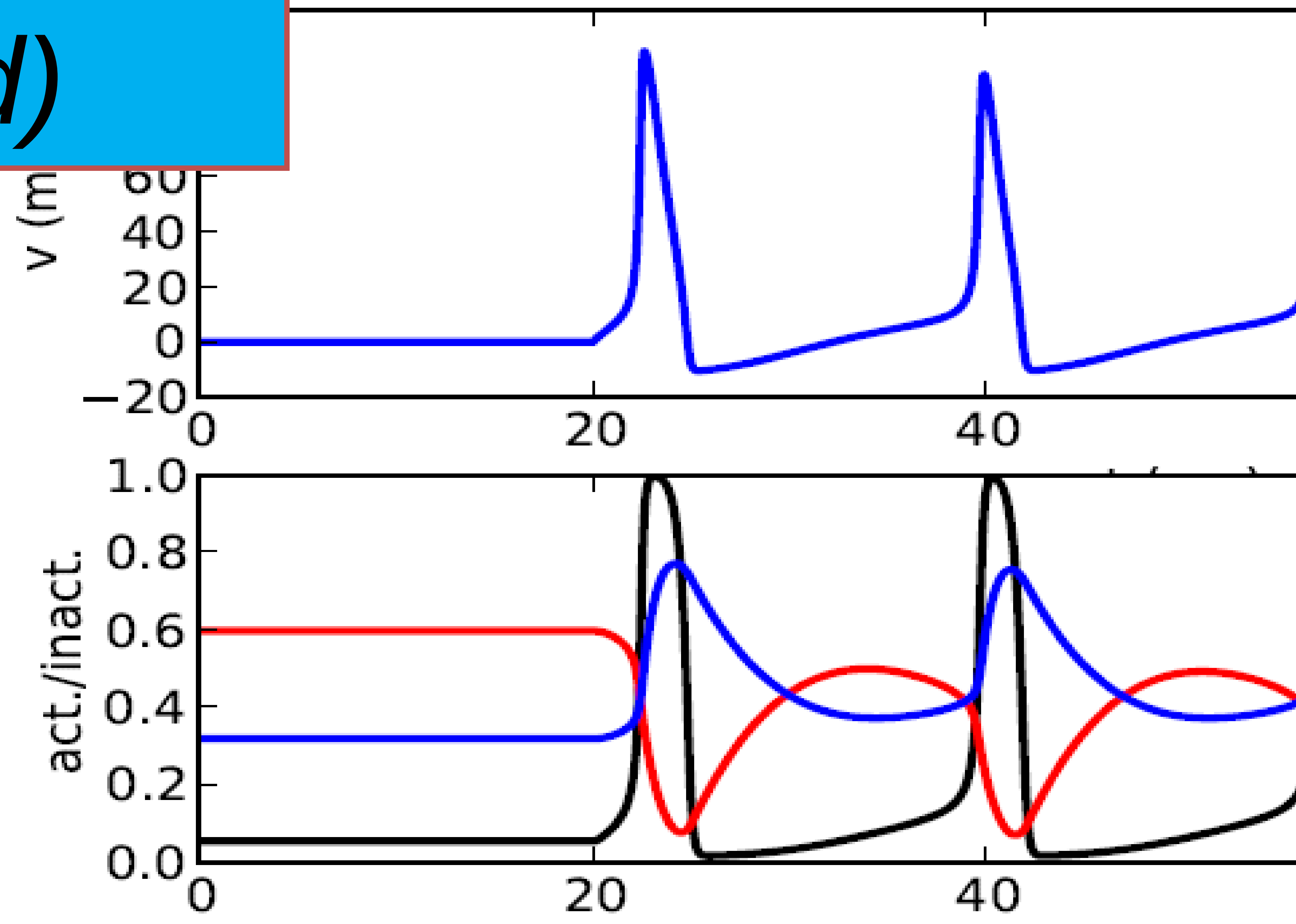
dynamics of *h* and *n* are similar

$$1 - h(t) = a\, n(t)$$

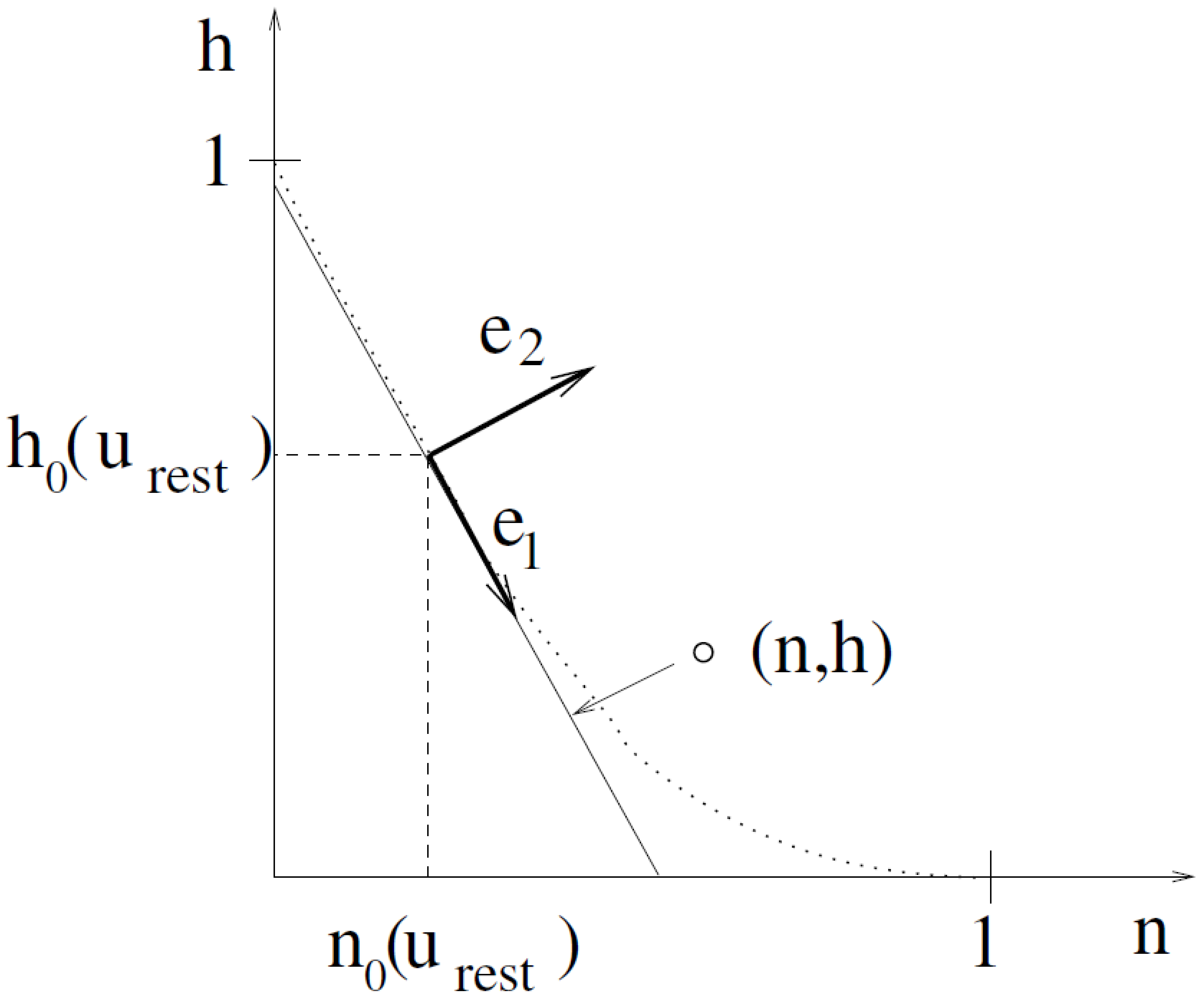
Math. Argument (blackboard)

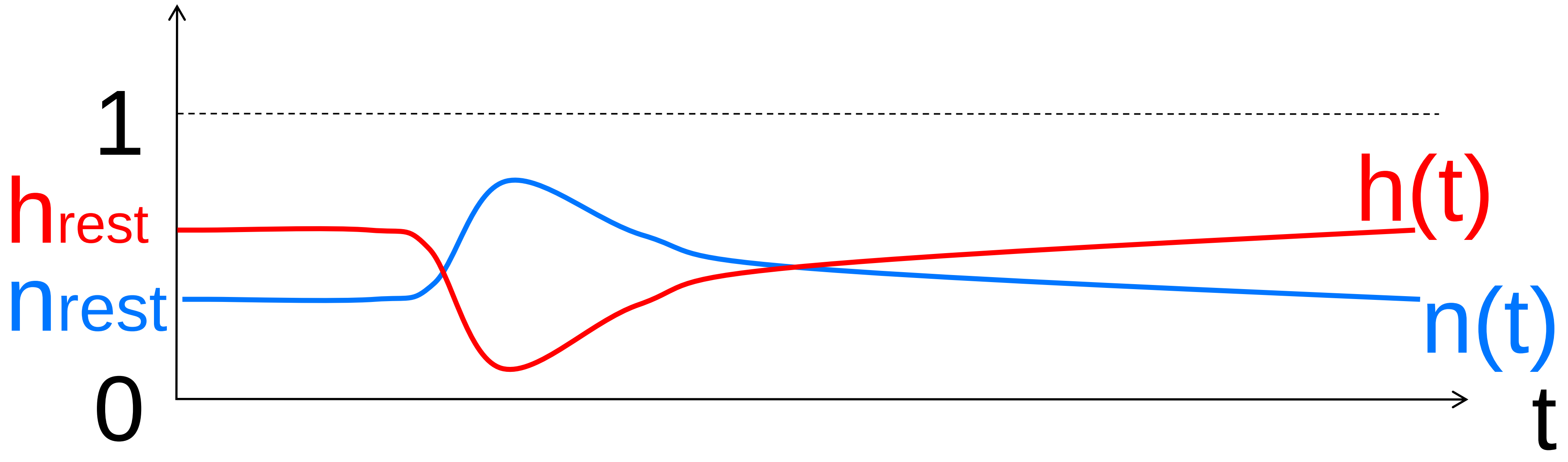






3.1 Detour 1. Exploit similarities/correlations





dynamics of *h* and *n* are similar

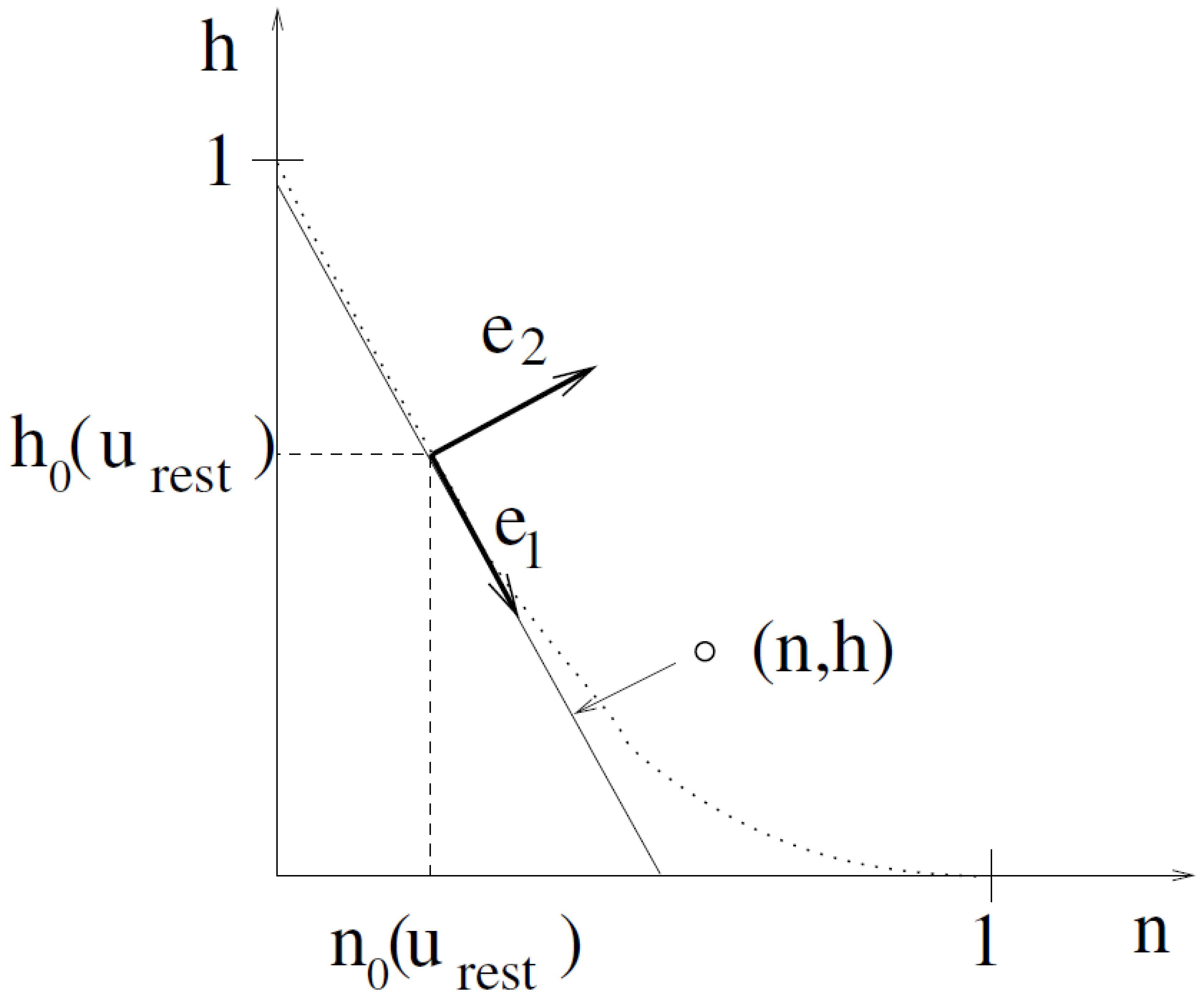
$$1 - h(t) = a\, n(t)$$

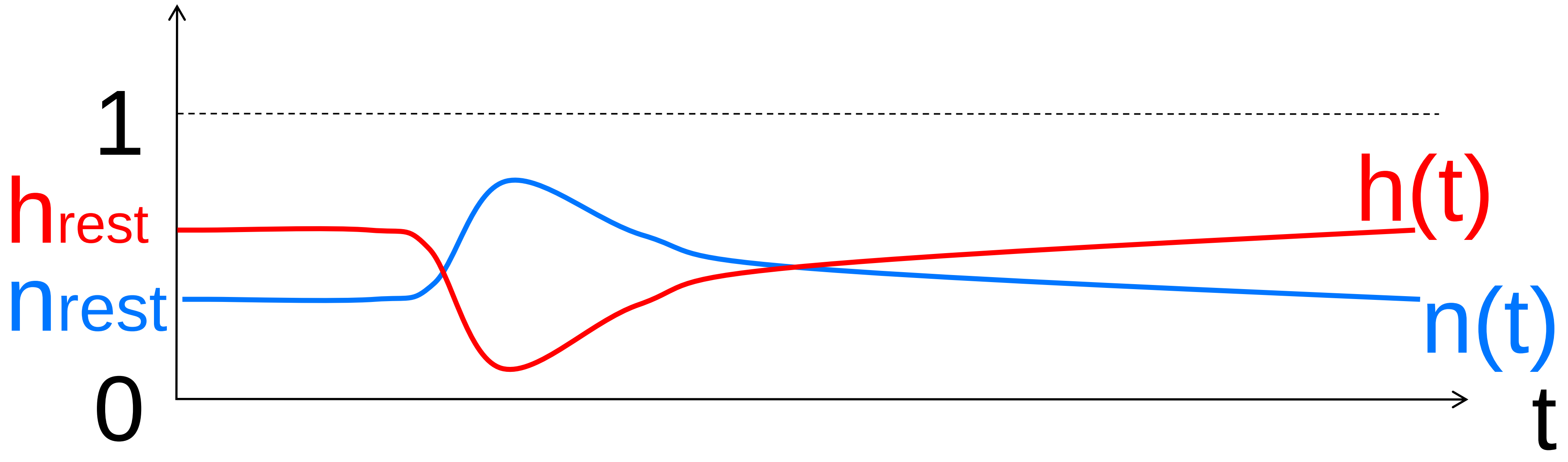
at rest

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n - n_0(u)}{\tau_n(u)}$$

3.1 Detour 1. Exploit similarities/correlations





dynamics of *h* and *n* are similar

(i) Rotate coordinate system
(ii) Suppress one coordinate
(iii) Express dynamics in new variable

$$1 - h(t) = a\, n(t) = w(t)$$

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n - n_0(u)}{\tau_n(u)}$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

3.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \underbrace{-g_{Na}[m(t)]^3 h(t)(u(t)-E_{Na})}_{I_{Na}} \; \underbrace{-g_K[n(t)]^4(u(t)-E_K)}_{I_K} \; \underbrace{-g_l(u(t)-E_l)}_{I_{leak}} + I(t)$$

$$C\frac{du}{dt} = -g_{Na}\, m_0(u)^3(1-w)(u-E_{Na}) - g_K[\frac{w}{a}]^4(u-E_K) - g_l(u-E_l) + I(t)$$

1) dynamics of *m* are fast $\longrightarrow$ $m(t) = m_0(u(t))$

2) dynamics of *h* and *n* are similar $\longrightarrow$ $\underbrace{1-h(t)}_{w(t)} = \underbrace{a\,n(t)}_{w(t)}$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n-n_0(u)}{\tau_n(u)}$$

$$\longrightarrow \frac{dw}{dt} = -\frac{w-w_0(u)}{\tau_{eff}(u)}$$

3.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \underbrace{-g_{Na}\, m_0(u)^3 (1-w)(u-E_{Na})}_{I_{Na}} \underbrace{- g_K (\frac{w}{a})^4 (u-E_K)}_{I_K} \underbrace{- g_l (u-E_l)}_{I_{leak}} + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

$$\tau \frac{du}{dt} = F(u(t), w(t)) + R\, I(t)$$

$$\tau_w \frac{dw}{dt} = G(u(t), w(t))$$

3.1. Reduction to 2 dimensions

2-dimensional equation

$$C\frac{du}{dt} = f(u(t), w(t)) + I(t)$$

$$\frac{dw}{dt} = g(u(t), w(t))$$

Enables graphical analysis! → Phase plane analysis

-Discussion of threshold
- Constant input current *vs* pulse input
-Type I and II
- Repetitive firing

Week 3 – Quiz 3.2-similar dynamics

Exploiting similarities:

A sufficient condition to replace two gating variables *r,s* by a single gating variable *w* is

[ ] Both *r* and *s* have the same time constant (as a function of u)

[ ] Both *r* and *s* have the same activation function

[ ] Both *r* and *s* have the same time constant (as a function of u) AND the same activation function

[ ] Both *r* and *s* have the same time constant (as a function of u) AND activation functions that are identical after some additive rescaling

[ ] Both *r* and *s* have the same time constant (as a function of u) AND activation functions that are identical after some multiplicative rescaling

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Week 3 – Reducing detail: Two-dimensional neuron models

3.1 **From Hodgkin-Huxley to 2D**

- Overview: From 4 to 2 dimensions
- MathDetour 1: Exploiting similarities
- MathDetour 2: Separation of time scales

3.2 **Phase Plane Analysis**

- Role of nullclines

3.3 **Analysis of a 2D Neuron Model**

- constant input vs pulse input
- MathDetour 3: Stability of fixed points

Wulfram Gerstner

EPFL, Lausanne, Switzerland

Discuss Exercise 1 – MathDetour 3.1: Separation of time scales

Exercise 1 (week 3)

even more general

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + h(y)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$$\tau_1 \ll \tau_2 \rightarrow x = h(y)$$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(h(y))$$

Discuss exercise 1 – Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = -\overbrace{g_{Na} m^3 h (u - E_{Na})}^{I_{Na}} \overbrace{- g_K n^4 (u - E_K)}^{I_K} \overbrace{- g_l (u - E_l)}^{I_{leak}} + \overset{\text{stimulus}}{I(t)}$$

$$\frac{dm}{dt} = -\frac{m - m_0(u)}{\tau_m(u)}$$

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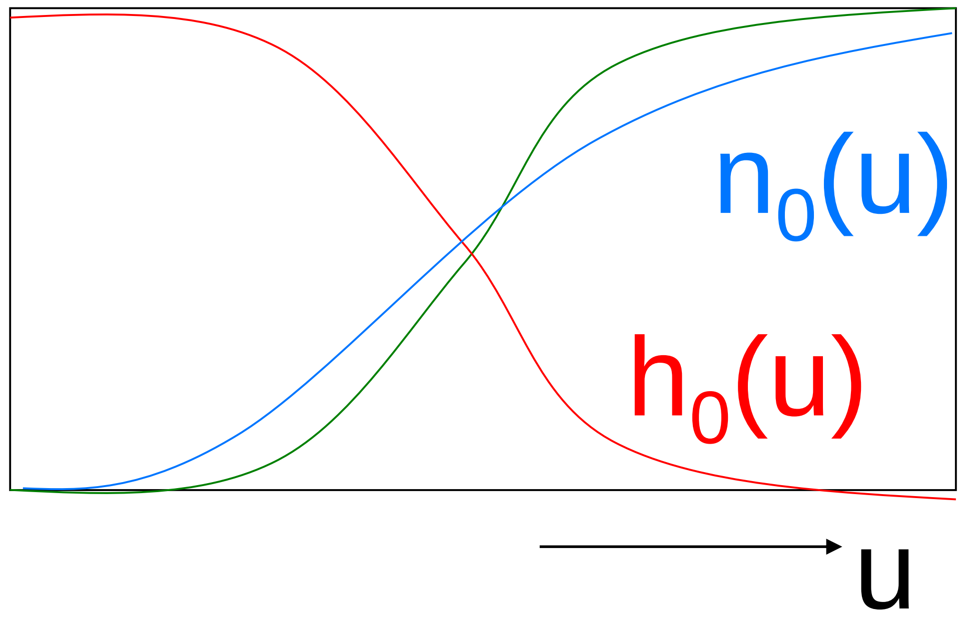

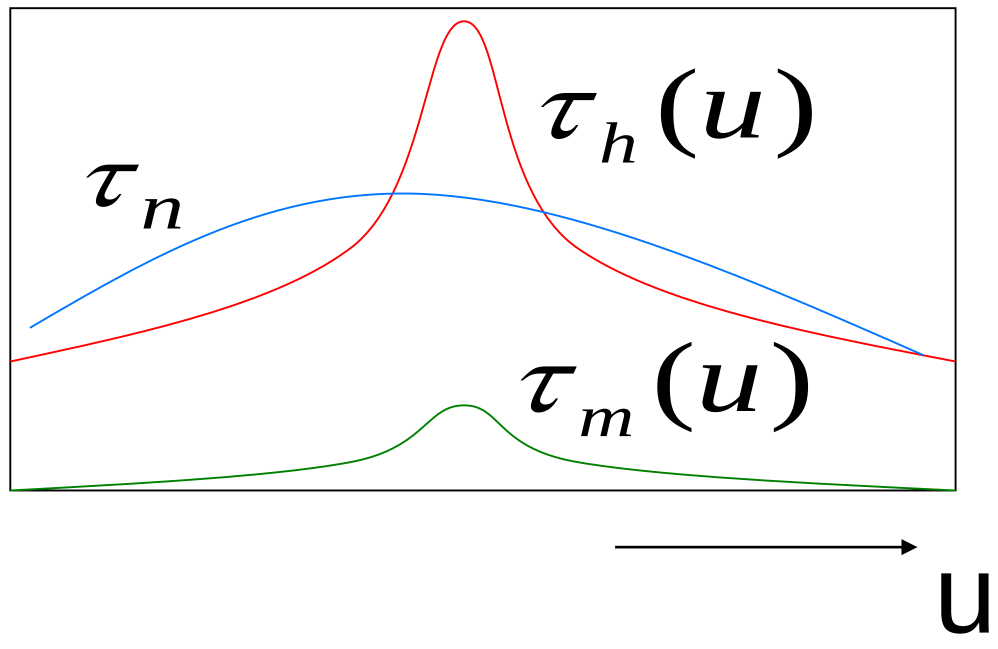

dynamics of *m* is fast $\longrightarrow$ $m(t) = m_0(u(t))$

Fast compared to what?

Neuronal Dynamics – Quiz 3.3.

A- Separation of time scales:
We start with two equations

$$\tau_1 \frac{dx}{dt} = -x + y + I(t)$$

$$\tau_2 \frac{dy}{dt} = -y + x^2 + A$$

[ ] If $\tau_1 << \tau_2$ then the system can be reduded to

$$\tau_2 \frac{dy}{dt} = -y + [y + I(t)]^2 + A$$

[ ] If $\tau_2 << \tau_1$ then the system can be reduded to

$$\tau_1 \frac{dx}{dt} = -x + x^2 + A + I(t)$$

[ ] None of the above is correct.

Pay attention to *I(t):*
We assume that I(t) is slow compared to both time constants.

Week 3 – Summary 3.1

In order to reduce the HH model from 4 to 2 equations we have to simplify. We use two different mathematical methods.

1. Separation of time scale.

If the time scale of two variables is different by a factor 10 or a hundred, we can assume that the faster one of the two variables has already converged to its 'momentary stable state' on the slow time scale. Thus, we can remove the fast variable. We use the separation of time scale to remove the variable m.

2. Exploit similarities.

If two variables evolve on the same time scale, they have, if we are lucky, some similar temporal evolution. We can reduce the two variables to one dimension by turning the coordinate system such that the first dimension is the one where the two variables evolve 'together'. The simplification consists in suppressing the second variable. This is similar to PCA where you would also only keep the first component. However, we need to do this such that also the DYNAMICS stays approximately correct, after reduction to 1 dimension.
We use this trick to compress h and n into a single variable w.

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3.2 Phase Plane Analysis

- Role of nullclines

3.3 Analysis of a 2D Neuron Model

- constant input vs pulse input
- MathDetour 3: Stability of fixed points

3.2. Reduced Hodgkin-Huxley model

$$C\frac{du}{dt} = \underbrace{-g_{Na}\, m_0(u)^3(1-w)(u-E_{Na})}_{I_{Na}} \underbrace{- g_K(\frac{w}{a})^4(u-E_K)}_{I_K} \underbrace{- g_l(u-E_l)}_{I_{leak}} + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_w(u)}$$

stimulus

$$\tau\frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w\frac{dw}{dt} = G(u,w)$$

3.2. Phase Plane Analysis/nullclines

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis!

-Discussion of threshold

-Type I and II

First step:

u-nullcline:

all points with *du/dt=0*

w-nullcline:

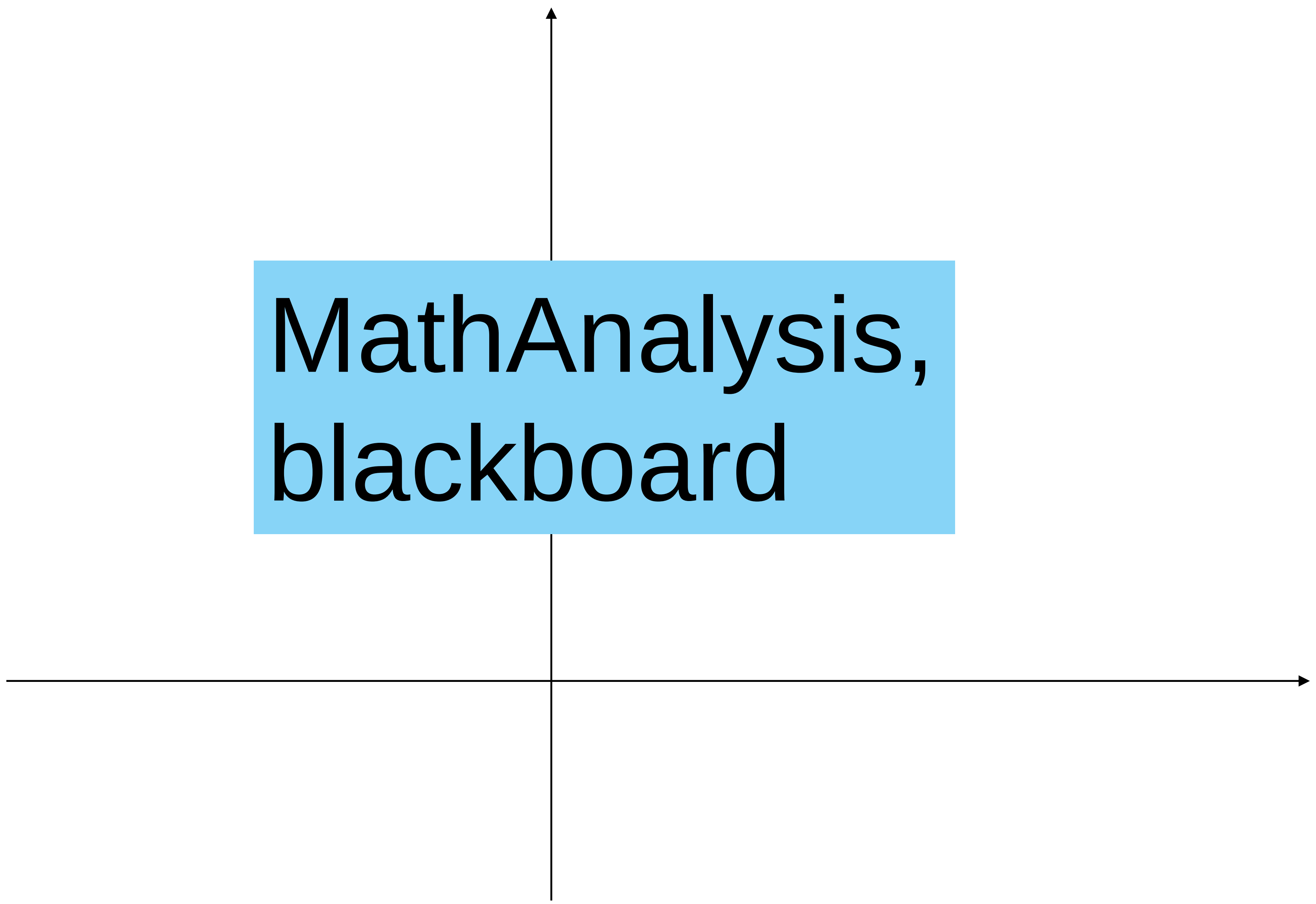
all points with *dw/dt=0*

3.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$= u - \frac{1}{3}u^3 - w + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$



u-nullcline

w-nullcline

3.2. flow arrows

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

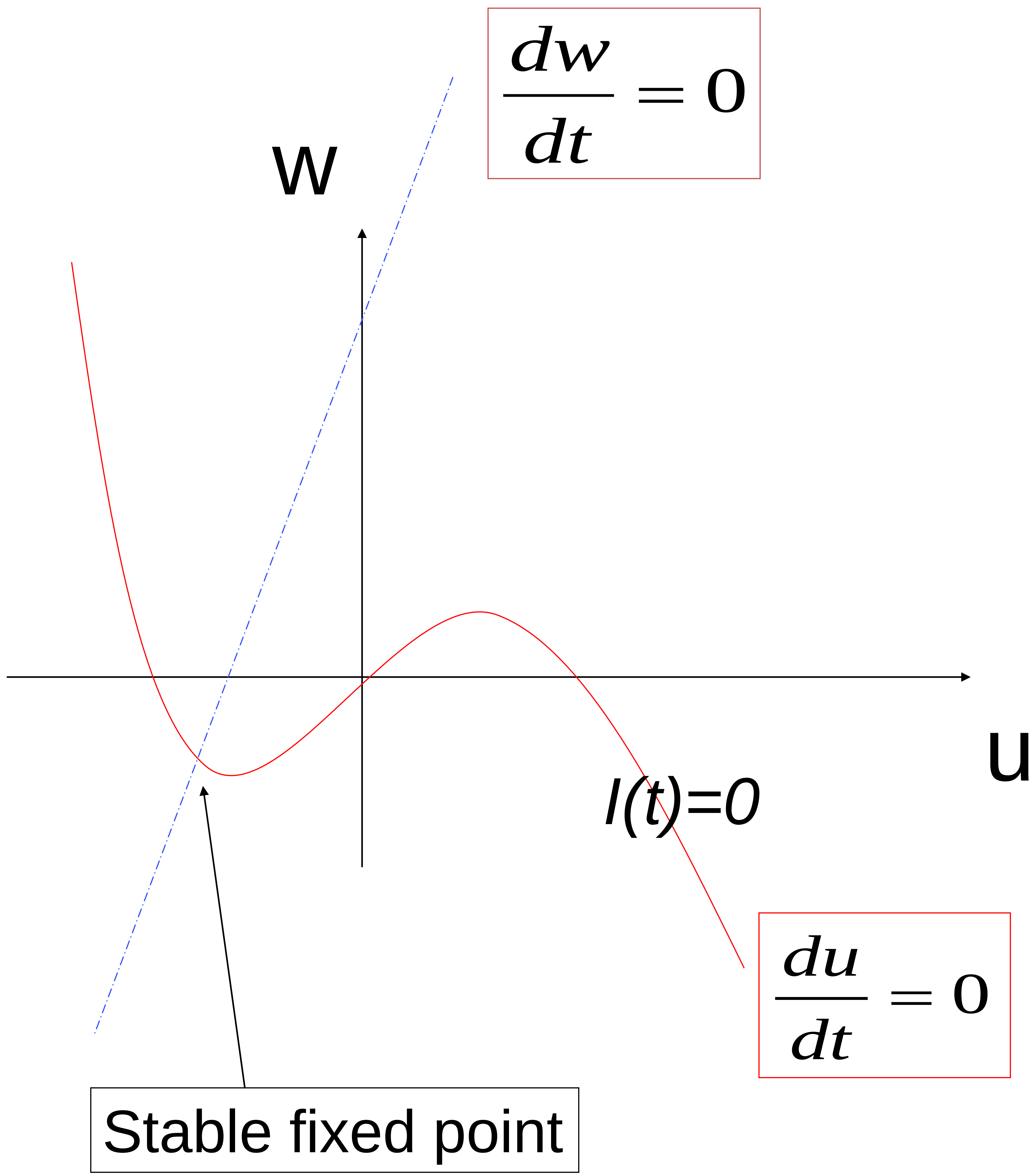
Stimulus I=0

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Consider change in small time step

Flow on nullcline

Flow in regions between nullclines



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Neuronal Dynamics – 3.2. flow arrows

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

Stimulus I=0

$$\tau_w \frac{dw}{dt} = G(u,w)$$

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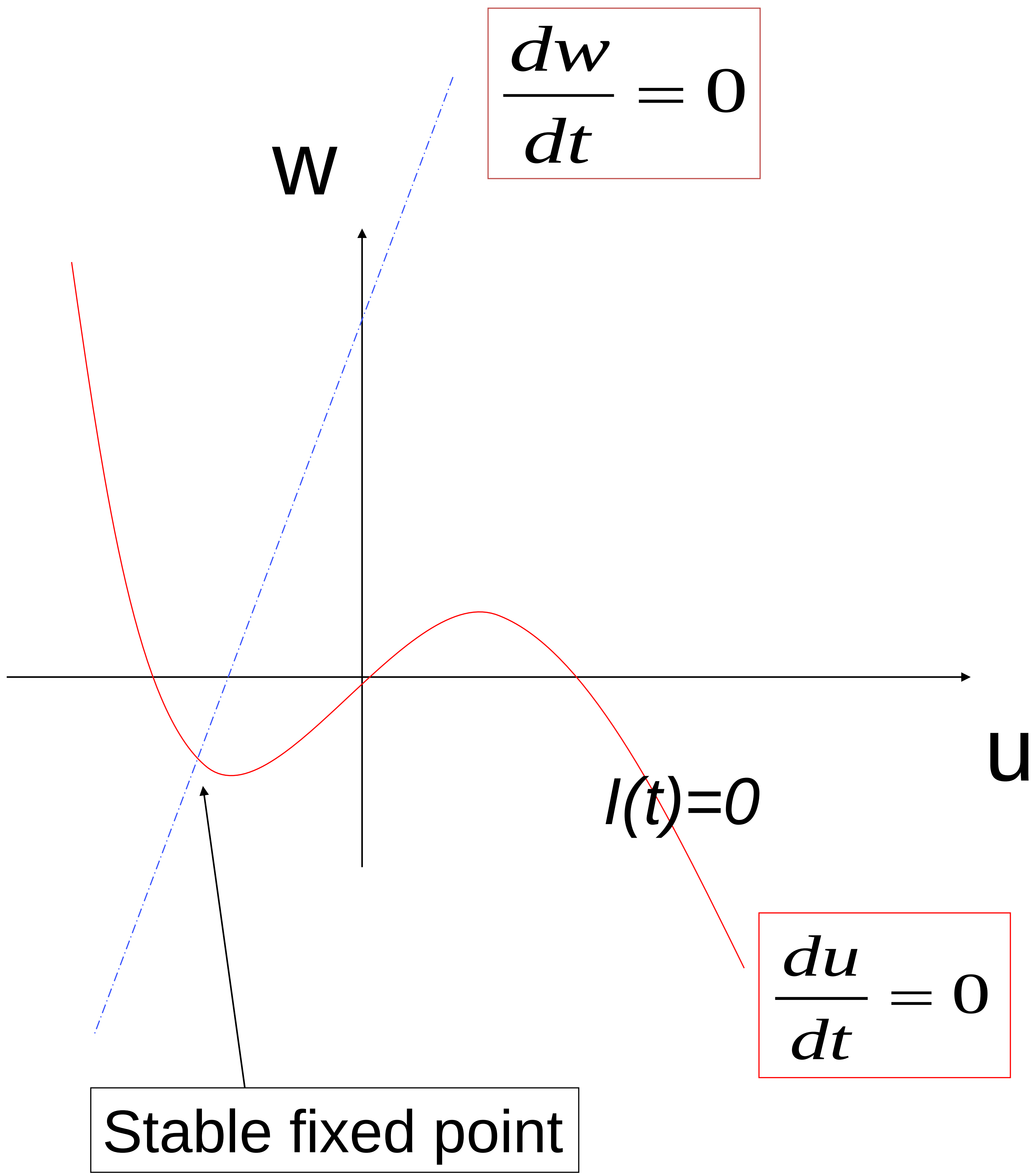


Week 3 – Quiz 3.4

Take 1 minute

A. u-Nullclines

[ ] On the u-nullcline, arrows are always vertical
[ ] On the u-nullcline, arrows point always vertically upward
[ ] On the u-nullcline, arrows are always horizontal

B. w-Nullclines

[ ] On the w-nullcline, arrows are always vertical
[ ] On the w-nullcline, arrows are always horizontal
[ ] On the w-nullcline, arrows point always to the left

Exercise 1: Separation of time scales

Now Ex 1 and 2.1
Lecture at 11h15

A. One-dimensional system

Consider the following differential equation

$$\tau \frac{dx}{dt} = -x + c\,. \tag{1}$$

1.1 Find the fixed point x_0 of this system. Hint: a fixed point is a stationary solution $\Rightarrow \frac{dx}{dt} = 0$.

1.2 Show that the fixed point is a stable one, and that the solution of (1) converges exponentially towards the fixed point with a time constant τ. Hint: write down the solution assuming an initial condition $x(t=0) \neq x_0$.

1.3 Consider the case where c is time-dependent, namely,

$$c \equiv c(t) = \begin{cases} 0 & \text{for } t < 0 \\ c_0 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } t > 1\,. \end{cases}$$

Calculate the solution $x(t)$ with initial condition $x(t=-10) = 0$.

1.4 Take the expression $x(t)$ you have found in the previous question. Consider $\tau = 0.5$ and $\tau = 0.01$ and sketch the function graph.

B. Separation of time scales

Consider the following system of equations:

$$\begin{aligned} \frac{du}{dt} &= f(u) - m \\ \epsilon\frac{dm}{dt} &= -m + c(u) \end{aligned}$$

with $\epsilon = 0.01$.

1.5 Exploit the fact that $\epsilon \ll 1$ and reduce the system to one equation (note the similarity between the m-equation and Eq.(1)).

1.6 Set $f(u) = -au + b$ where $a > 0$, $b \in \mathbb{R}$ and $c(u) = \tanh(u)$. Discuss the stability of the fixed points with respect to a and b. Hint: use the graphical analysis for one dimensional equations from week 1: when plotting $f(u)$ and $c(u)$ against u, you can read off the fixed point from that graph.

Exercise 2: Phase plane stability analysis

2.1 Linear system

Consider the following linear system:

$$\begin{cases} \dfrac{du}{dt} &= \alpha u - w \\ \dfrac{dw}{dt} &= \beta u - w\,. \end{cases}$$

These equations can be written in matrix form as $\frac{d}{dt}x = Ax$ where $x = \begin{pmatrix} v \\ w \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & -1 \\ \beta & -1 \end{pmatrix}$. Determine the conditions for stability of the point $(u=0, w=0)$ in the case $\beta > \alpha$ by studying the eigenvalues of the above matrix. (Hint: Distinguish the cases of real and complex eigenvalues.)

2.2 Piecewise linear Fitzhugh-Nagumo model

The Fitzhugh-Nagumo model is defined by the equations

$$\begin{cases} \dfrac{du}{dt} &= F(u,w) = f(u) - w + I \\ \dfrac{dw}{dt} &= G(u,w) = bu - w \end{cases}$$

Here, $u(t)$ is the membrane potential and $w(t)$ is a second, time-dependent variable. I stands for the injected current. A simplified model is obtained by considering a piecewise linear $f(u)$:

$$f(u) = \begin{cases} -u & \text{if } u < 1 \\ \dfrac{u-1}{a} - 1 & \text{if } 1 \leq u < 1+2a \\ 2(1+a) - u & \text{if } u > 1+2a \end{cases}$$

with $a < 1$, $b > 1/a$.

(i) Sketch the “nullclines” $du/dt = 0$ and $dw/dt = 0$ in a (u,v)-plot. Consider the case $I = 0$. How does the fixed point move as I is varied? Sketch the form of the flow (i.e., the vector $(du/dt, dw/dt)$) along the nullclines and deduce qualitatively the shape of the trajectories.

(ii) Calculate the Jacobian matrix evaluated at the fixed point,

$$J = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial w} \end{pmatrix}.$$

Determine, by studying the eigenvalues of J, the linear stability of the fixed point as a function of I. What happens when the fixed point destabilizes?

3.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$= u - \frac{1}{3}u^3 + RI(t) \text{ -}w$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

change b_0 , b_1

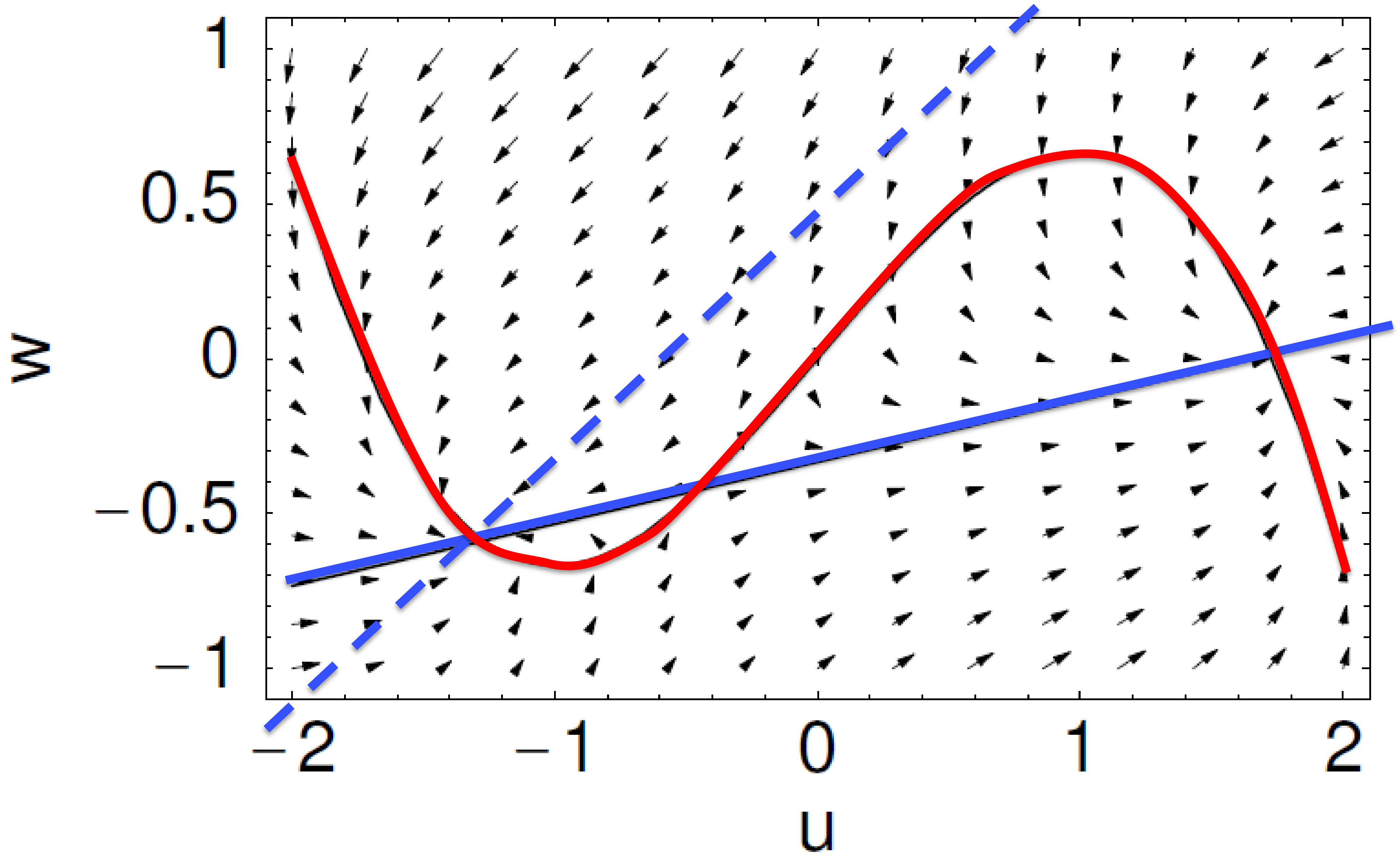


Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

3.2. Nullclines of reduced HH model

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

u-nullcline

w-nullcline

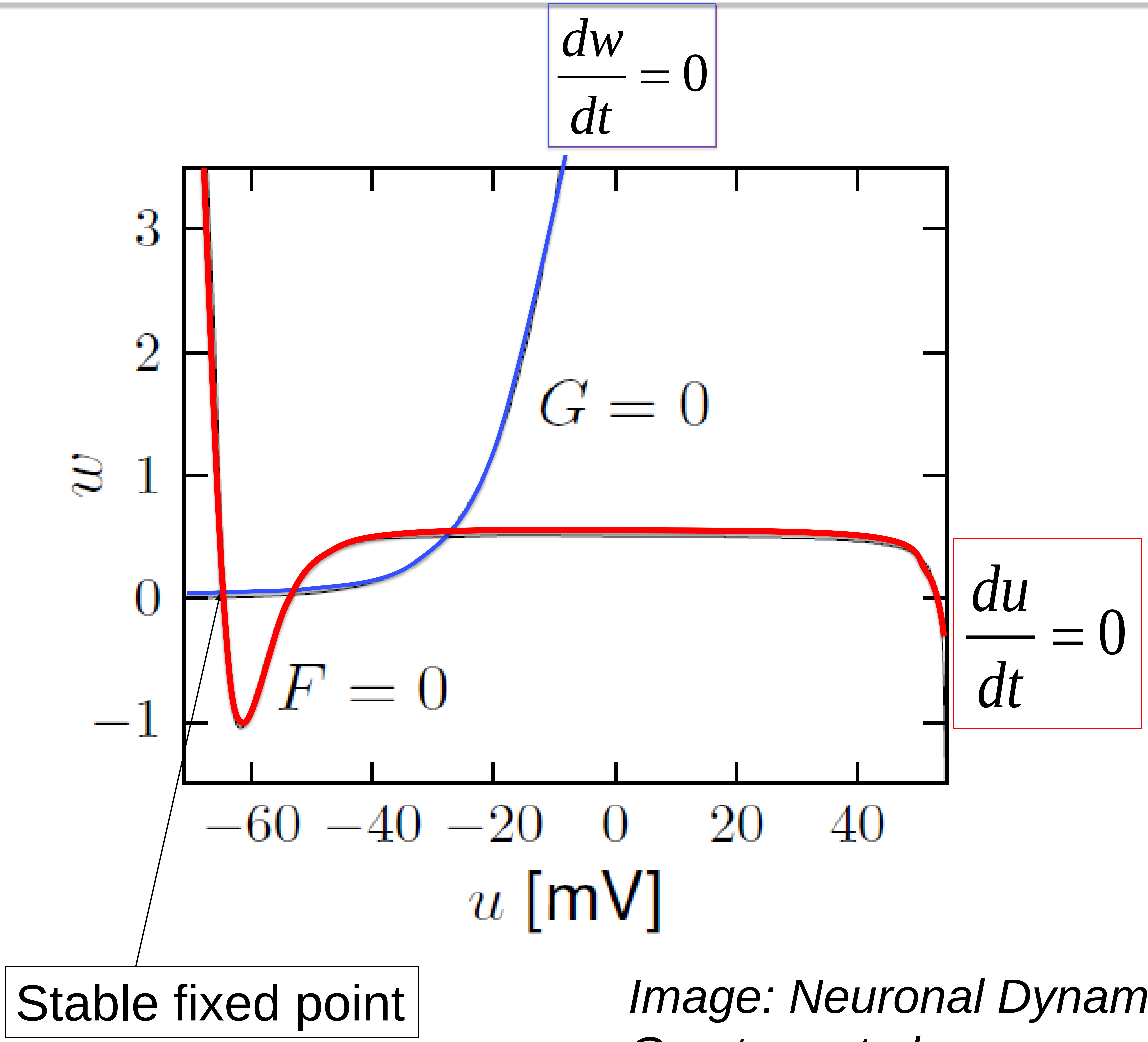


Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

3.2. Phase Plane Analysis

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis! $\longrightarrow$ Application to neuron models

Important role of

- nullclines
- flow arrows

Week 3 – Summary 3.2

Once we are in two dimensions we can use phase plane analysis. Two important concepts are the ‘nullclines’; and the local direction of the ‘flow’.

Intersections of the two nullclines correspond to fixed points. It is a bit of work to decide whether a fixed point is stable or not. However, in some cases (such as a saddle point) stability is visible directly from the graph.

Stability of a fixed point is determined by linearizing around the fixed point. Since we are in 2 dimensions, linearization yields a 2x2 matrix. The eigenvalues determine the stability (See exercise 2.1).

The FitzHuhg Nagumo model is a particularly simple 2-dimensional model. The reduction of the full Hodgkin-Huxley model yields a more complicated picture in the phase plane.

3.3. Analysis of a 2D neuron model

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

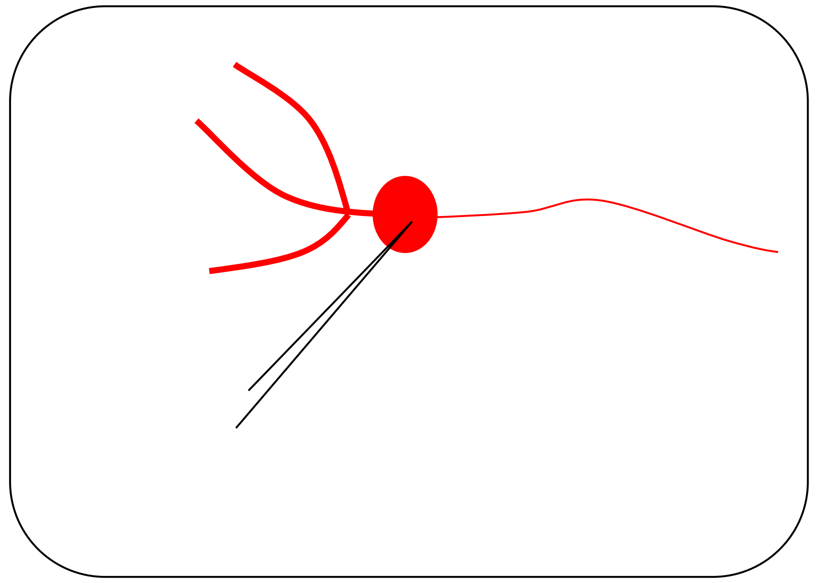
$$\tau_w \frac{dw}{dt} = G(u,w)$$

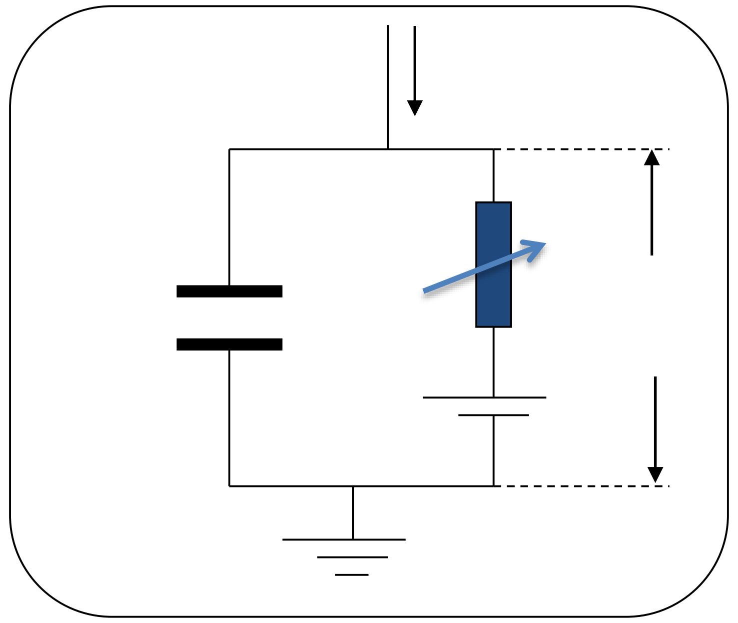
Enables graphical analysis!

2 important input scenarios

- Pulse input
- Constant input

3.3. 2D neuron model : Pulse input





$$\tau \cdot \frac{d}{dt} u = F(u,w) + RI$$

$$\tau_w \frac{d}{dt} w = G(u,w)$$

pulse input

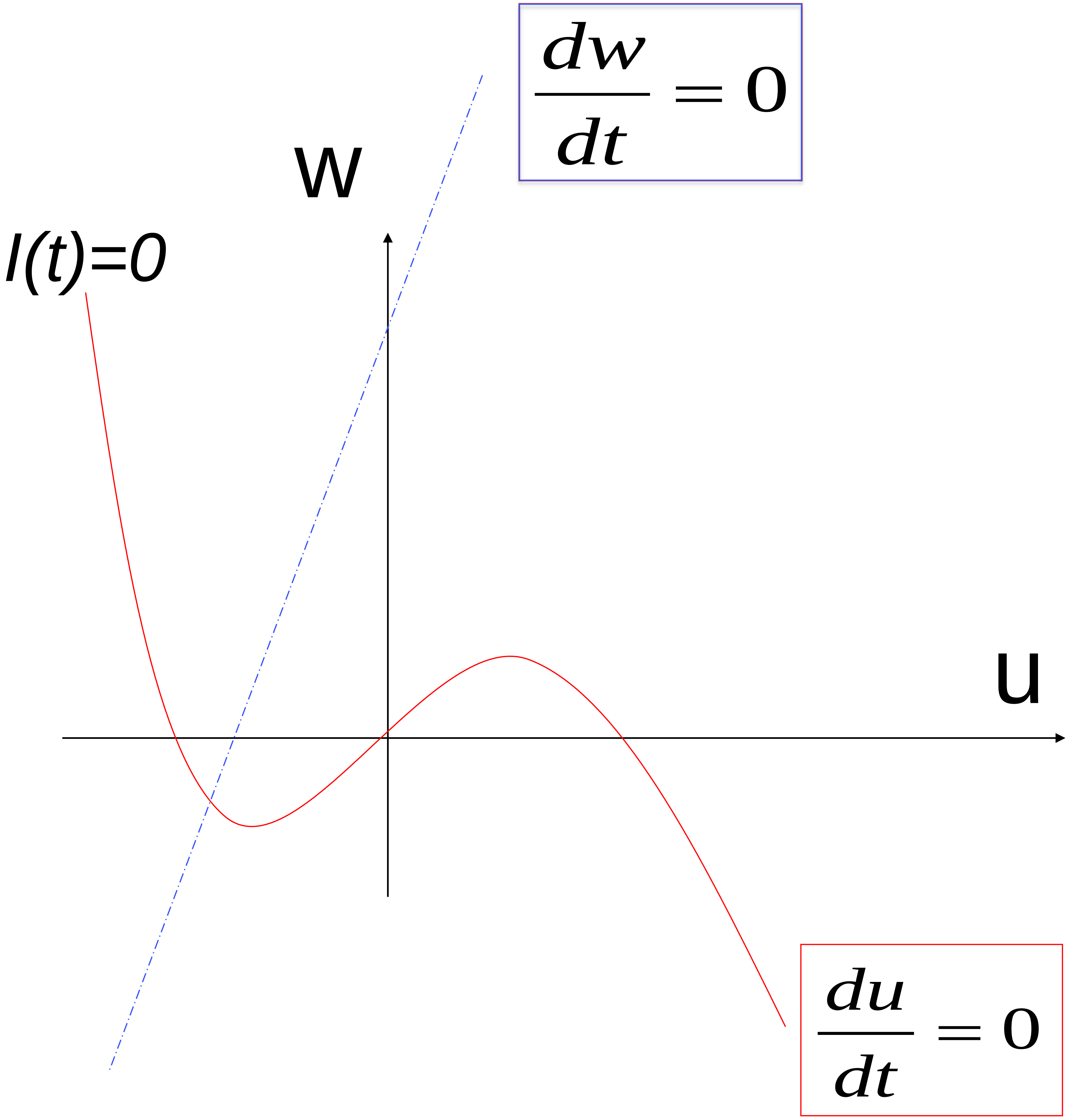
3.3. FitzHugh-Nagumo Model : Pulse input

$$\tau \frac{du}{dt} = F(u,w) + RI(t) = u - \frac{1}{3}u^3 - w + RI(t)$$

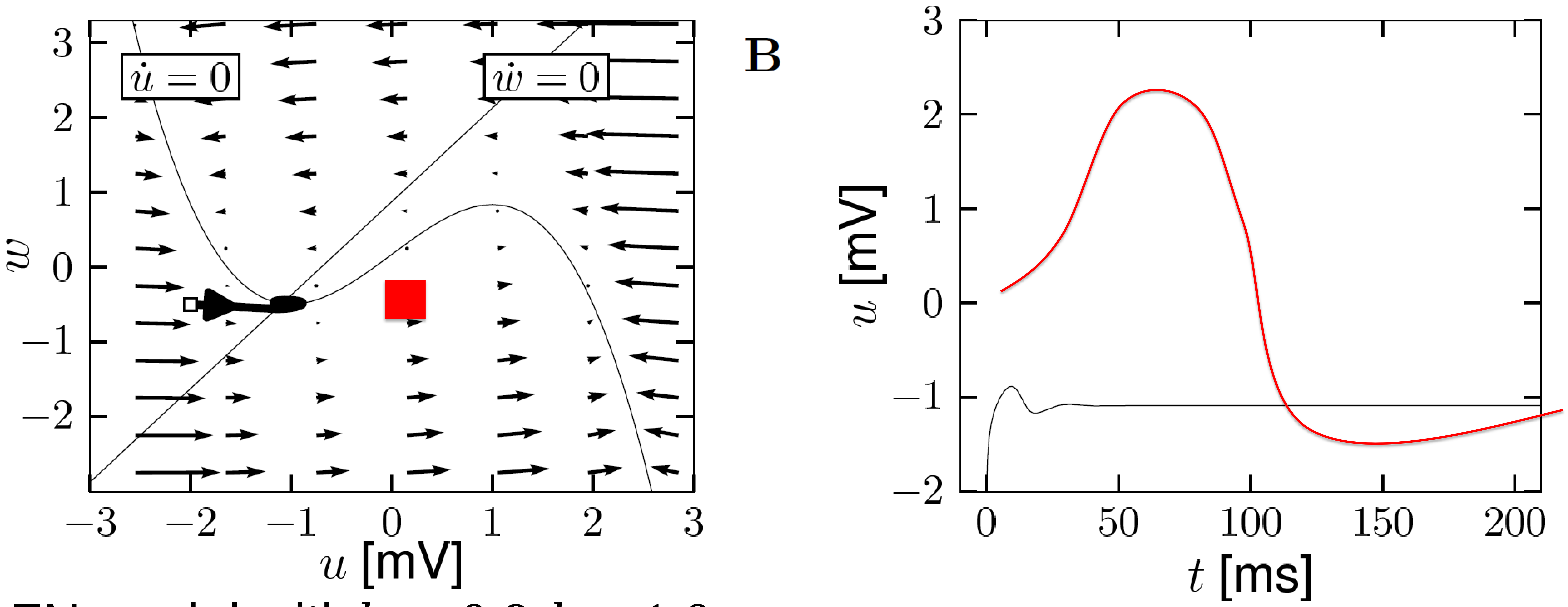
$$\tau_w \frac{dw}{dt} = G(u,w) \qquad = b_0 + b_1 u - w$$

pulse input

Pulse input: jump of voltage



3.3. FitzHugh-Nagumo Model : Pulse input



FN model with $b_0 = 0.9; b_1 = 1.0$

Pulse input: jump of voltage/initial condition

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

3.3. FitzHugh-Nagumo Model – 2 different inputs

Pulse input: DONE!

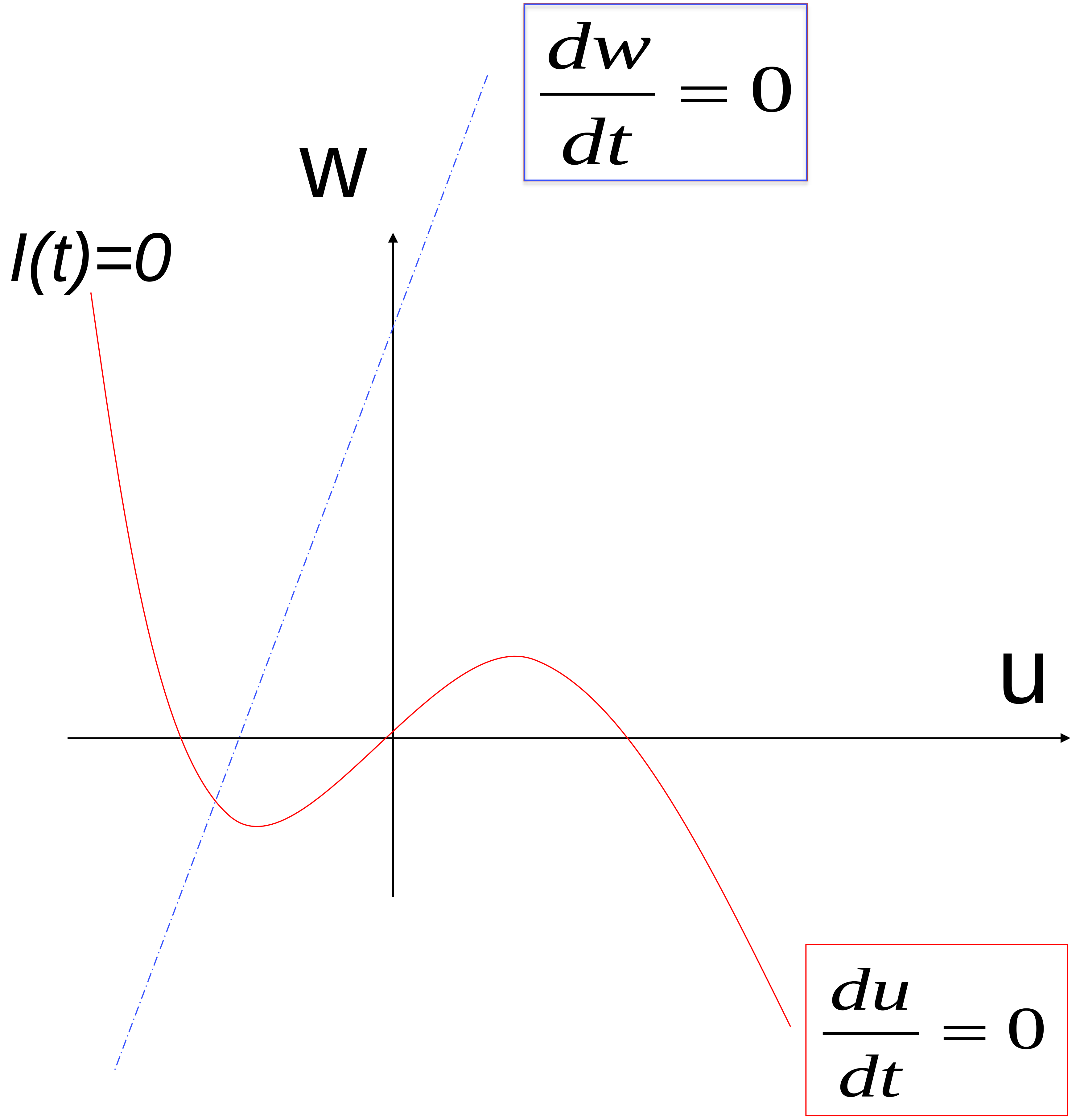
- jump of voltage
- 'new initial condition'
- spike generation for large input pulses

2 important input scenarios

constant input:

- graphics?
- spikes?
- repetitive firing?

Now



3.3. FitzHugh-Nagumo Model: Constant input

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

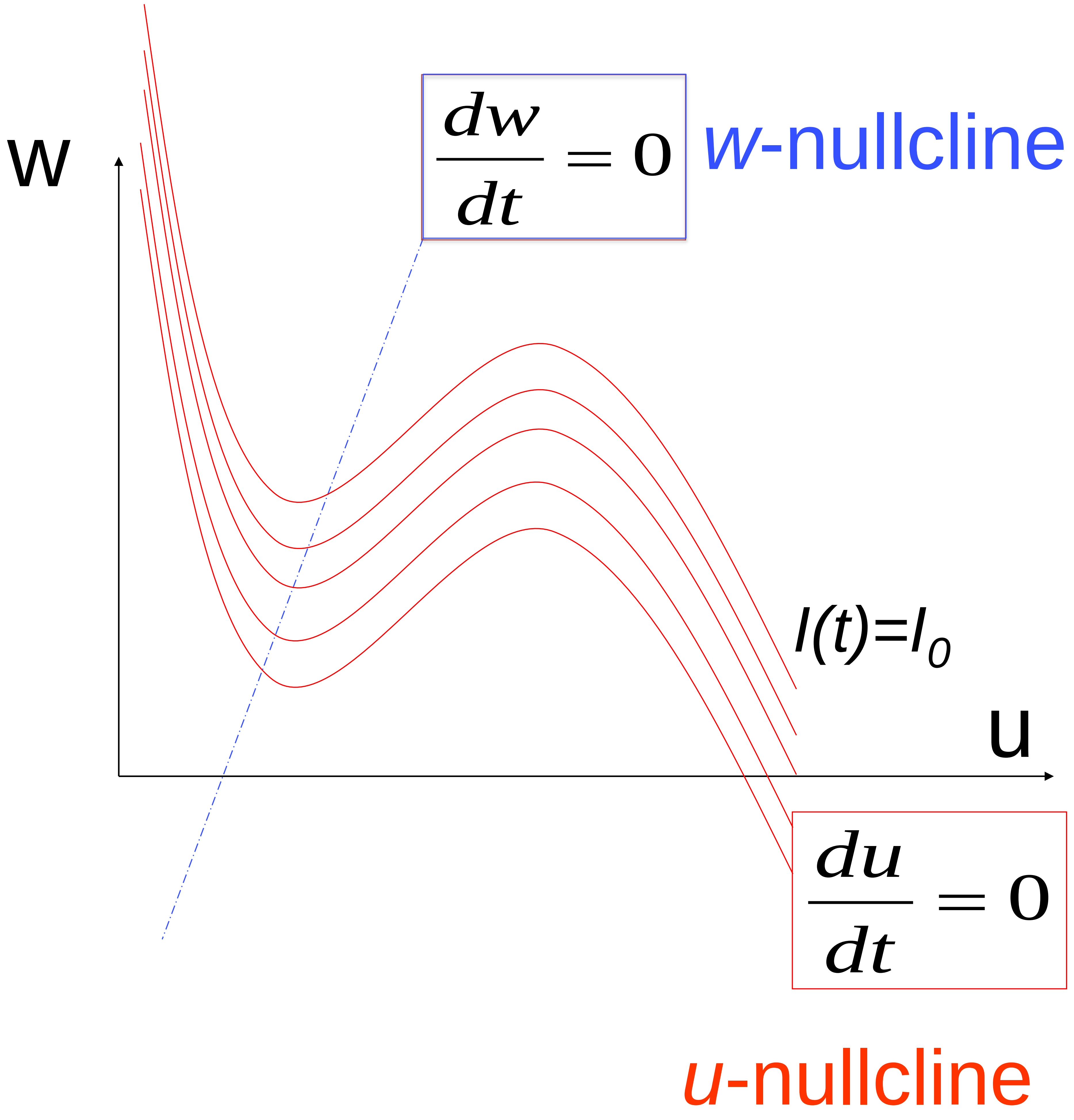
$$= u - \frac{1}{3}u^3 - w + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

Intersection point (fixed point)

-moves

-changes Stability



Week 3 – part 3: Analysis of a 2D neuron model

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Discussion of exercise 2 Detour. Stability of fixed points

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

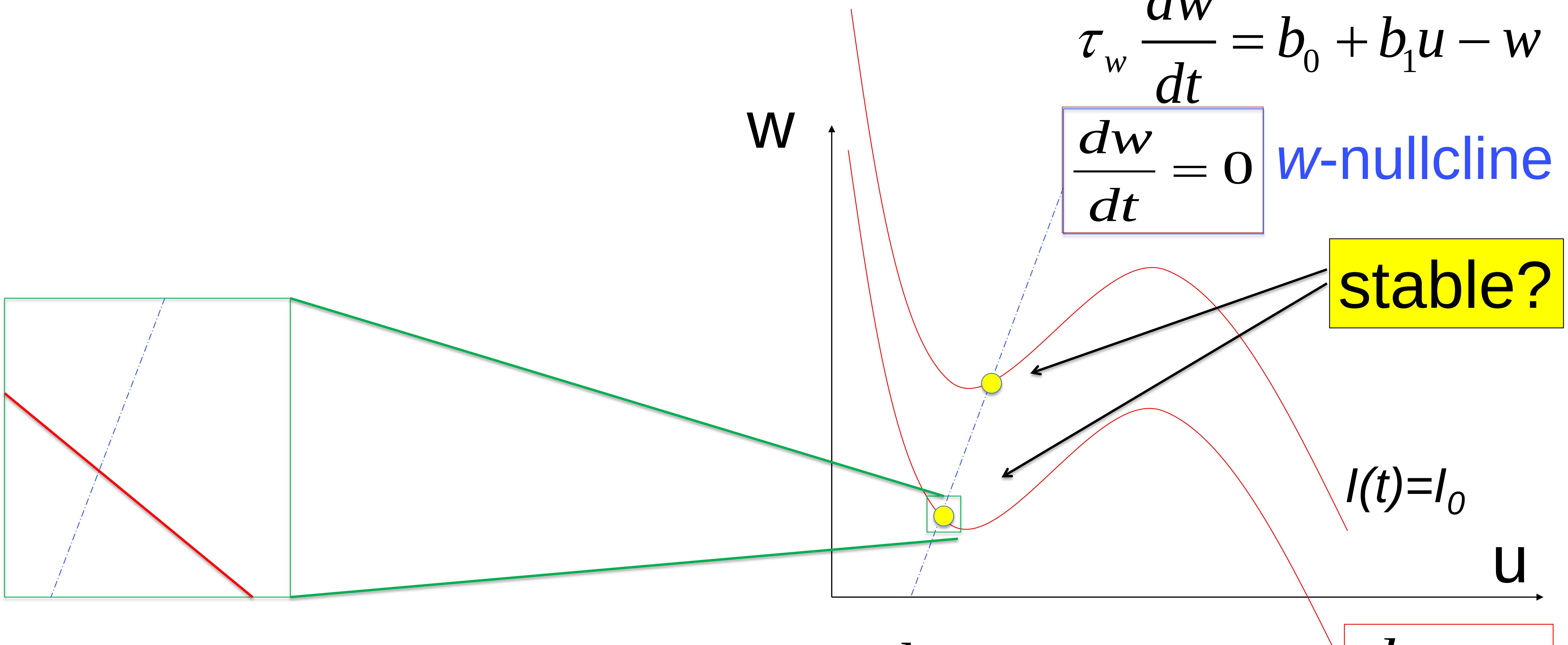
$$\tau_w \frac{dw}{dt} = G(u,w)$$

How to determine stability of fixed point?

Discussion of exercise 2 - Detour: Stability of fixed points.

$$\tau_w \frac{dw}{dt} = b_0 + b_1 u - w$$

$$\frac{dw}{dt} = 0$$ *w*-nullcline



$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\frac{du}{dt} = 0$$

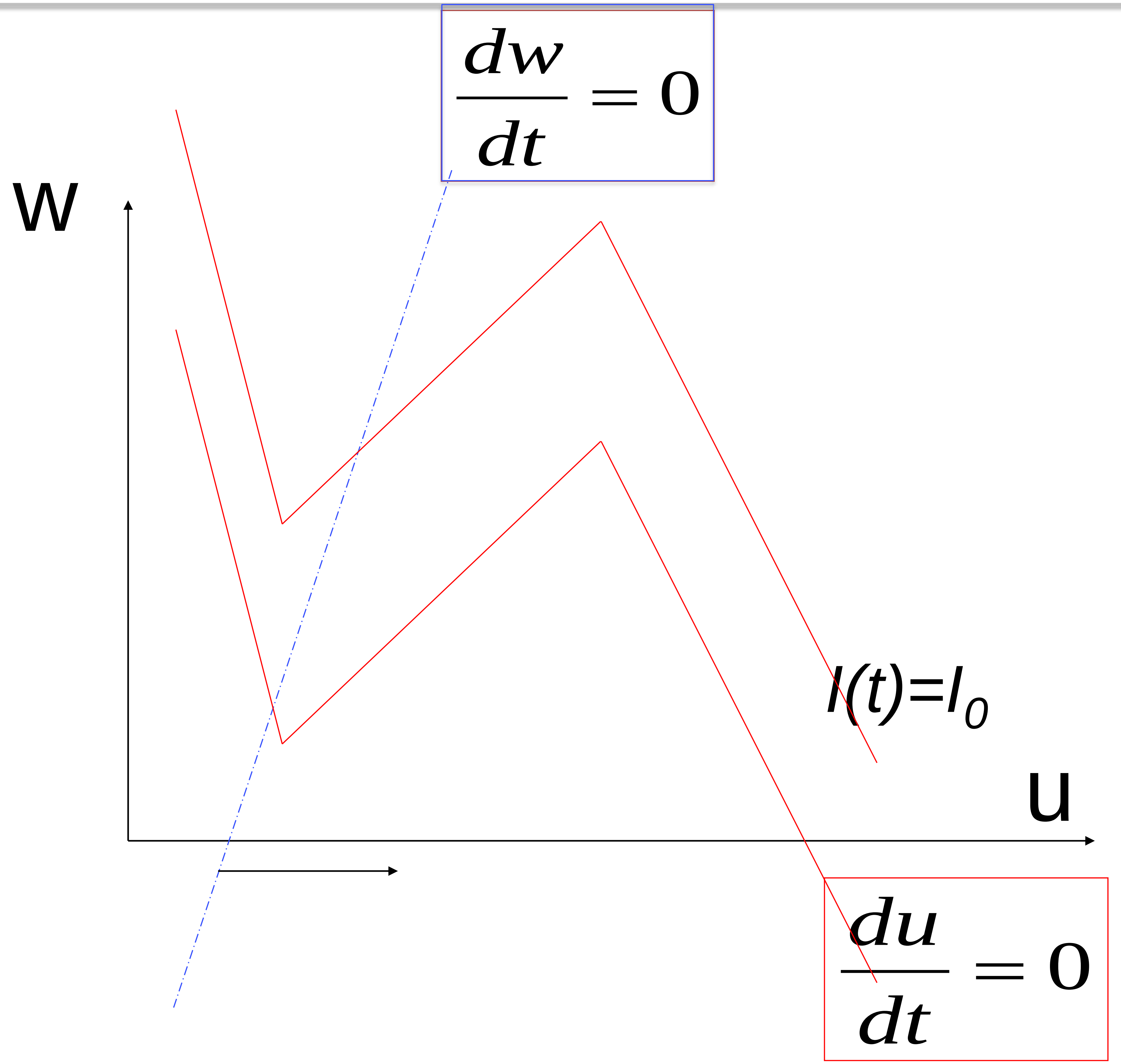
u-nullcline

Discussion of Exercise 2: Detour - Stability of fixed points

stimulus

$$\tau \frac{du}{dt} = au - w + I_0$$

$$\tau_w \frac{dw}{dt} = cu - w$$

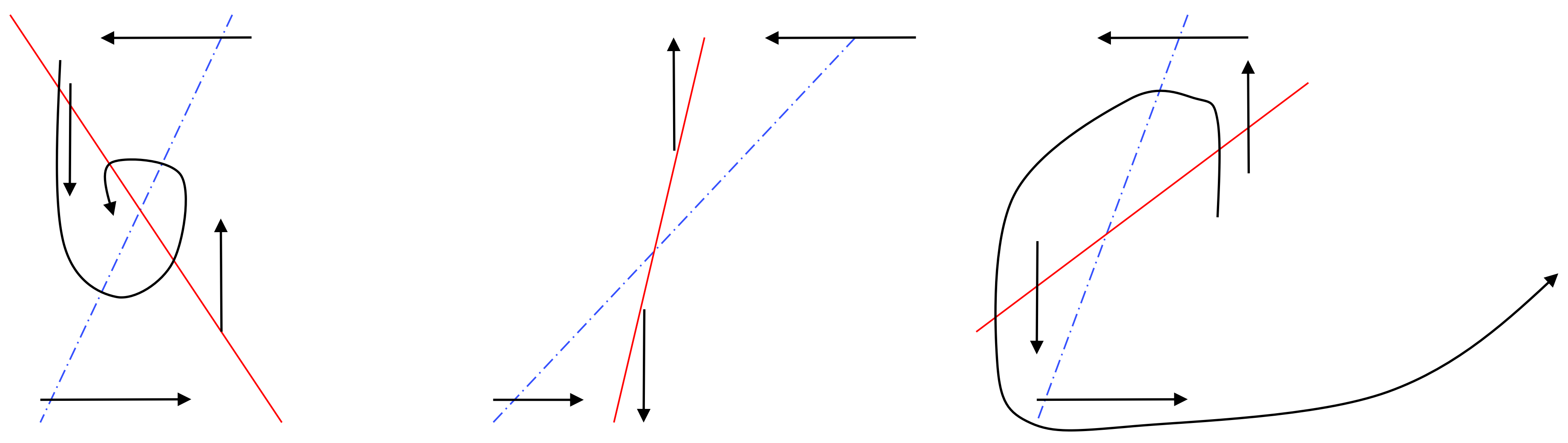


Discussion of Exercise 2: Detour. Stability of fixed points

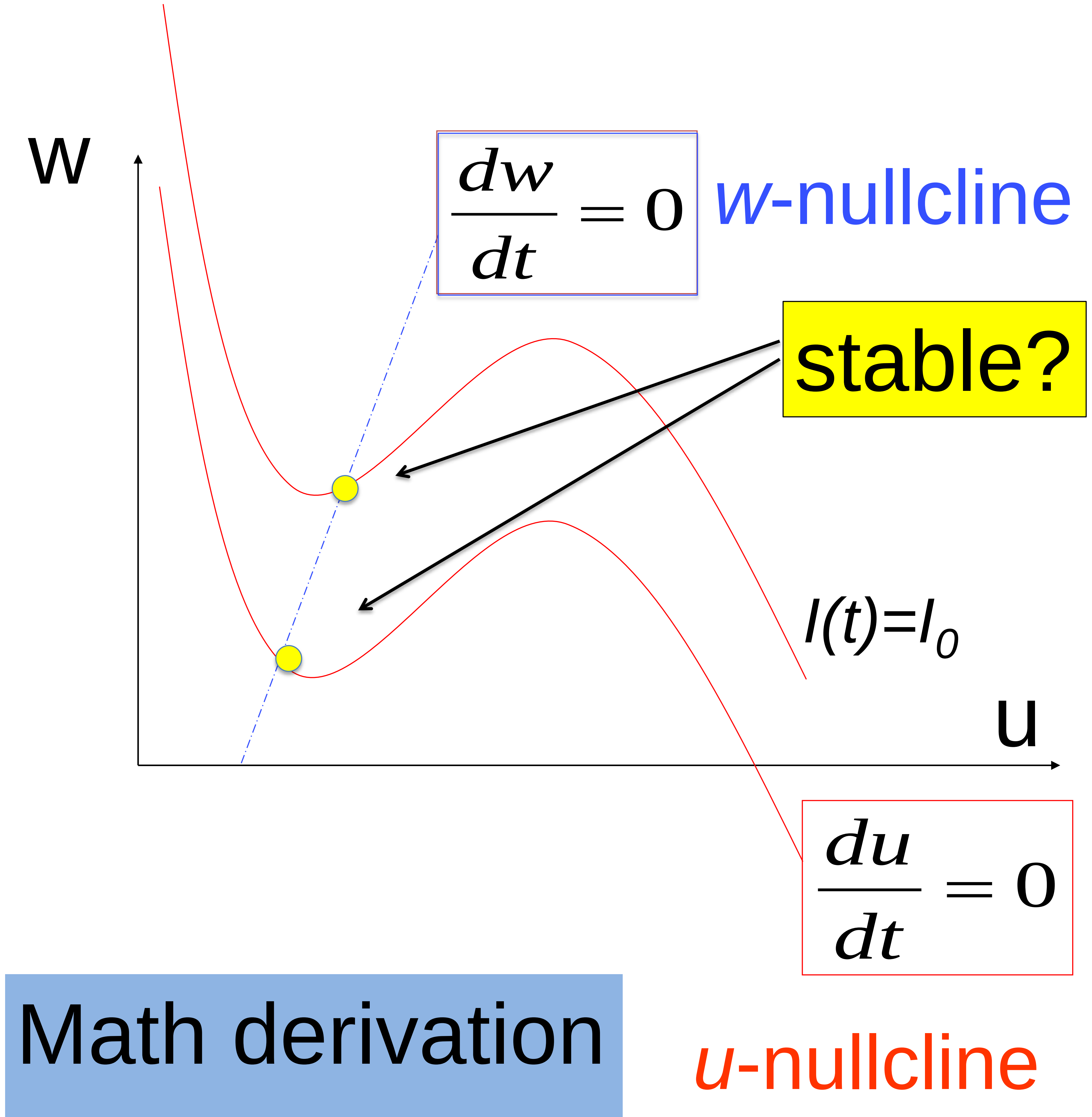
$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

zoom in:



stable saddle unstable



Math derivation now

3.3. Neuron models and Stability of fixed points

Now Back:

Application to our neuron model

2-dimensional equation

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Stability characterized by Eigenvalues of linearized equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x}$$

3.3. FitzHugh-Nagumo Model: Constant input

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

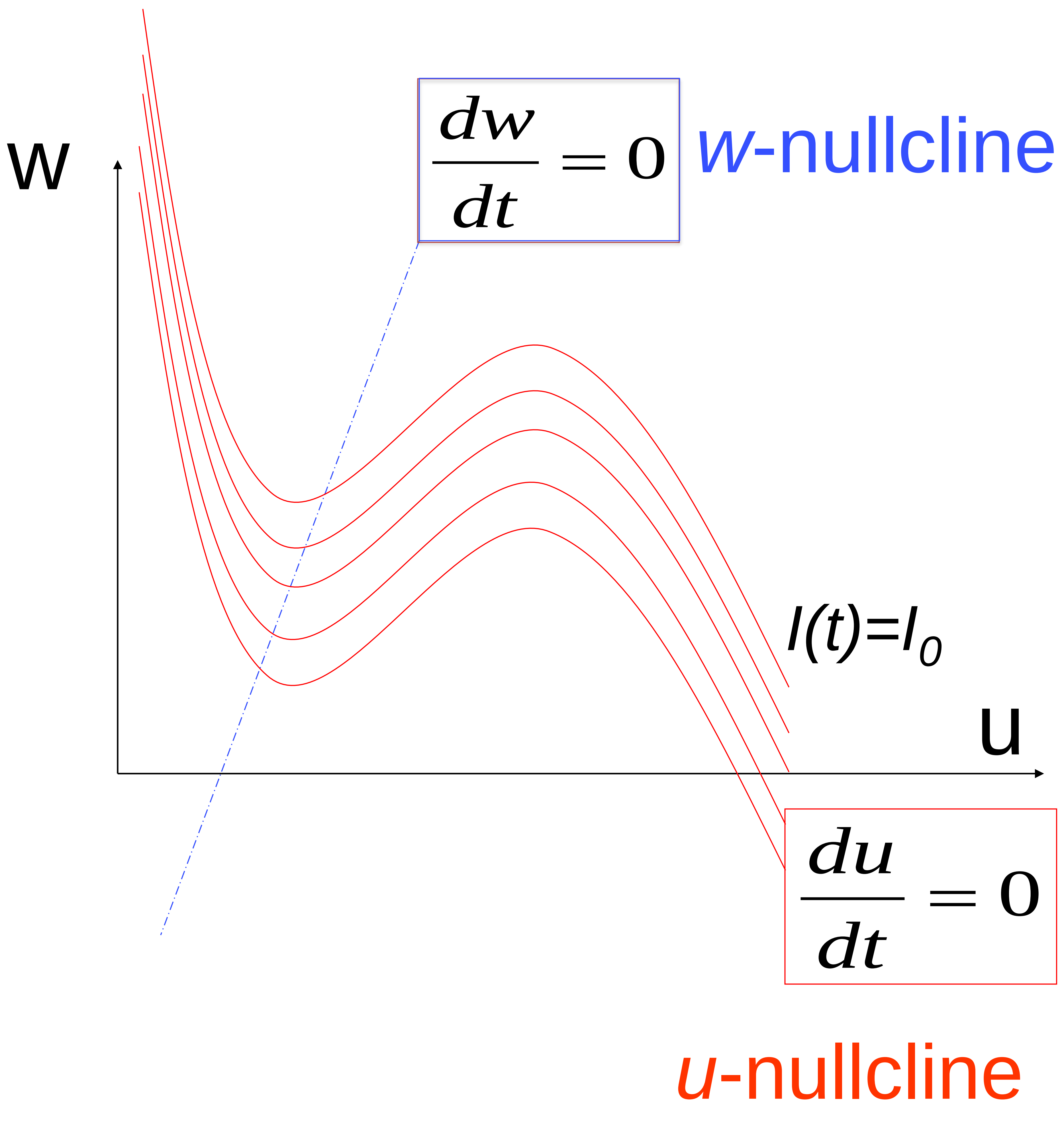
$$= u - \frac{1}{3}u^3 - w + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

Intersection point (fixed point)

-moves

-changes Stability



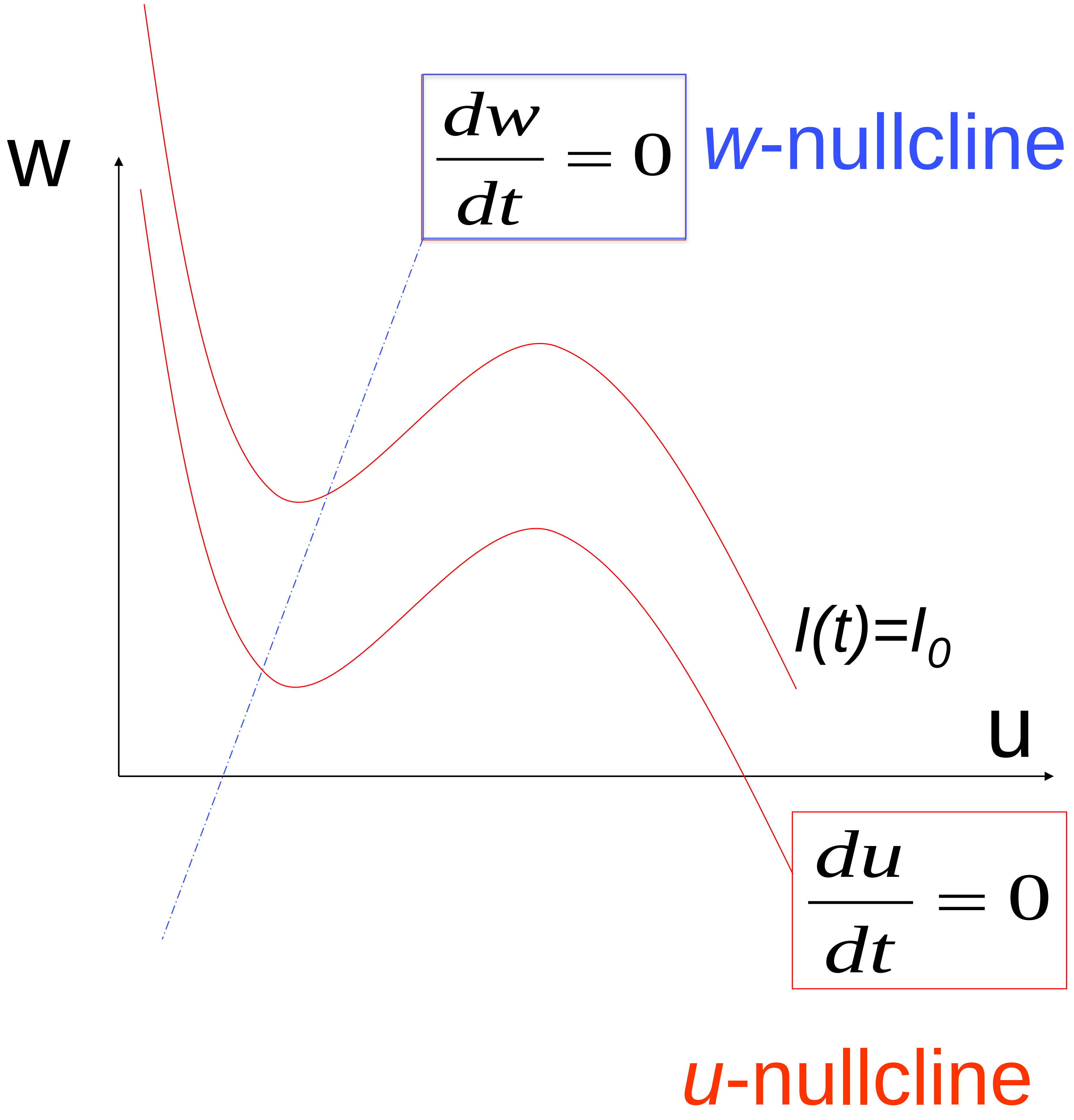
3.3. FitzHugh-Nagumo Model: Constant input

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

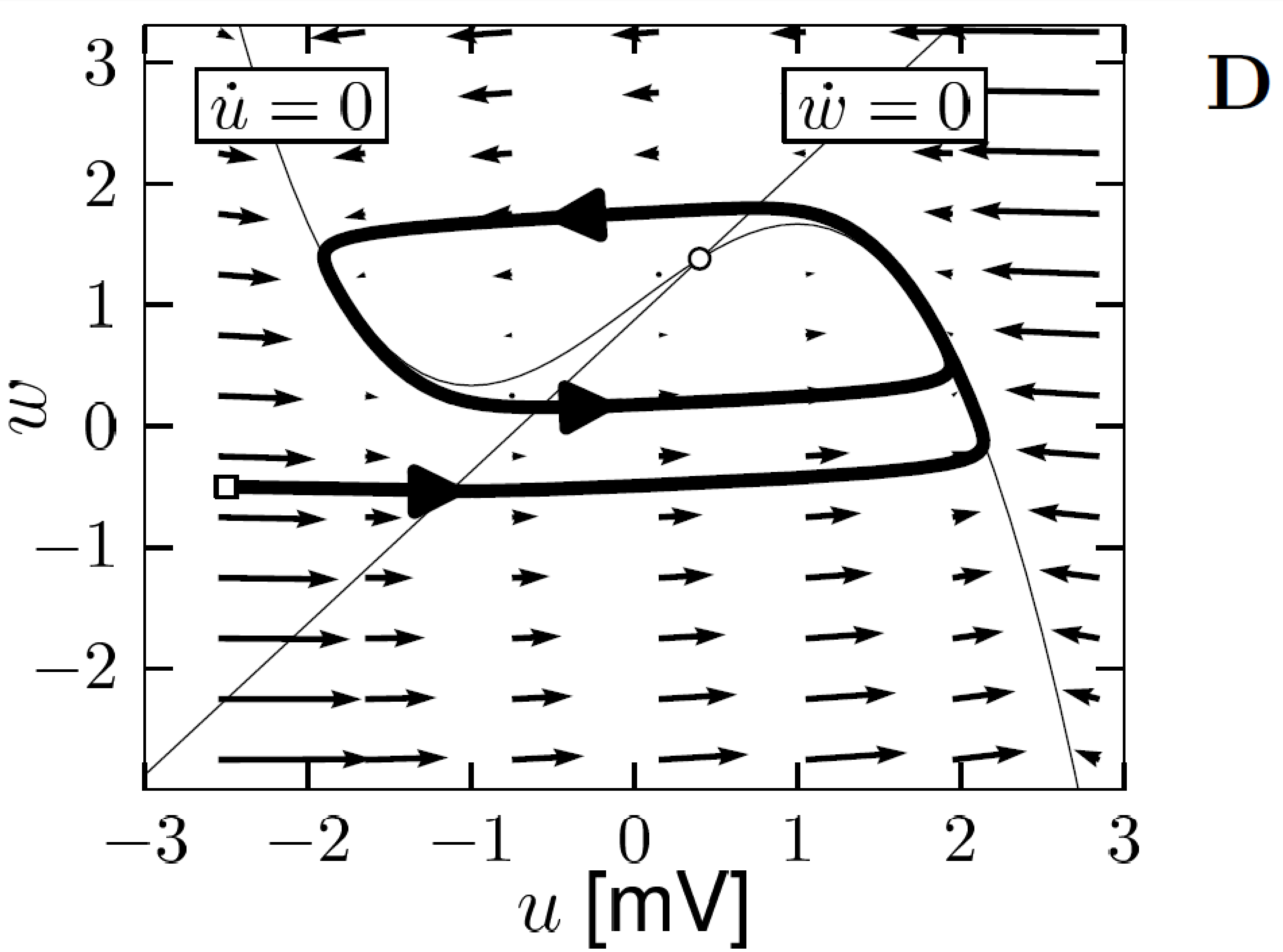
$$= u - \frac{1}{3}u^3 - w + RI_0$$

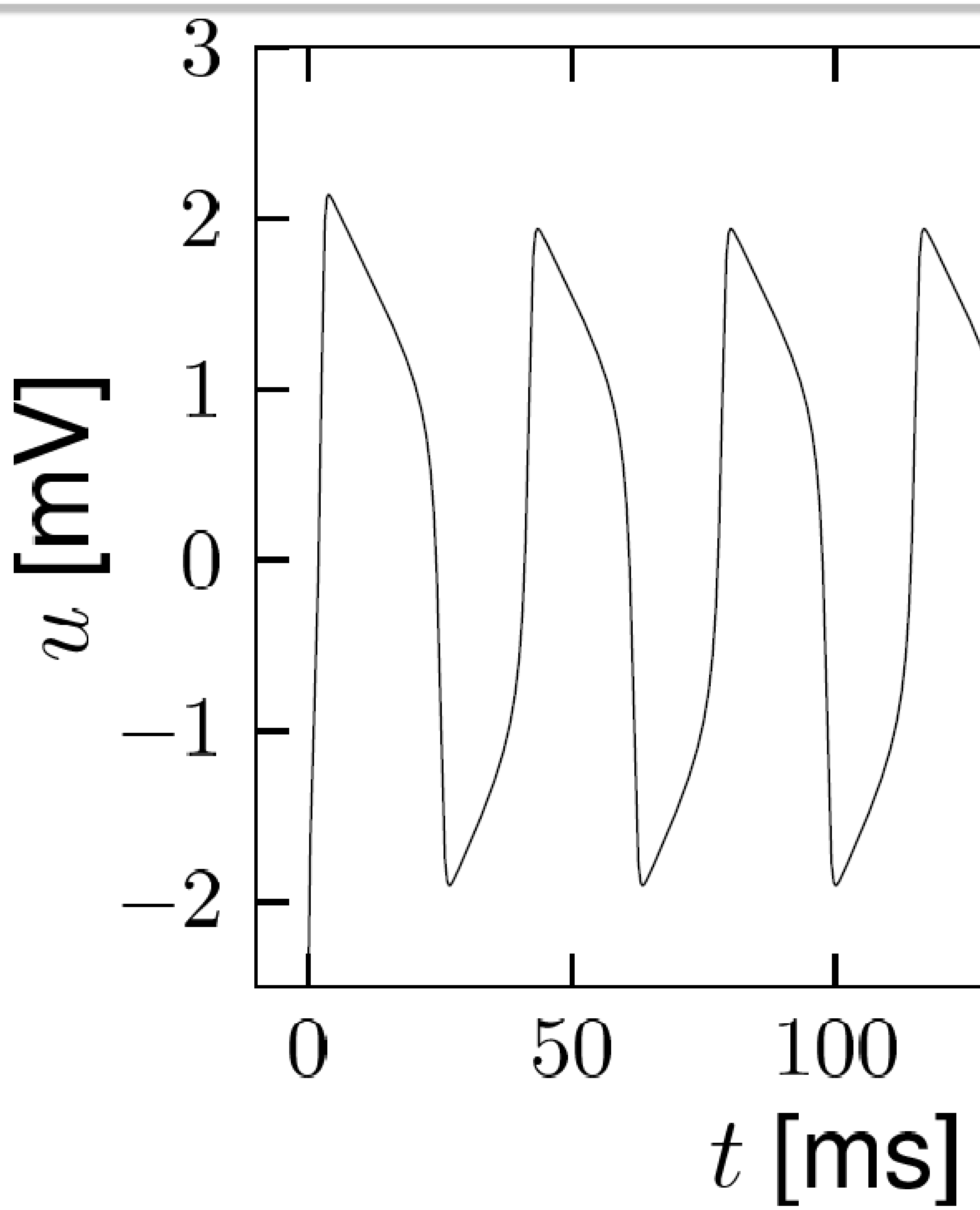
$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

Intersection point (fixed point)
-moves
-changes Stability



3.3. FitzHugh-Nagumo Model : Constant input





FN model with $b_0 = 0.9; b_1 = 1.0; RI_0 = 2$

constant input: u-nullcline moves

limit cycle

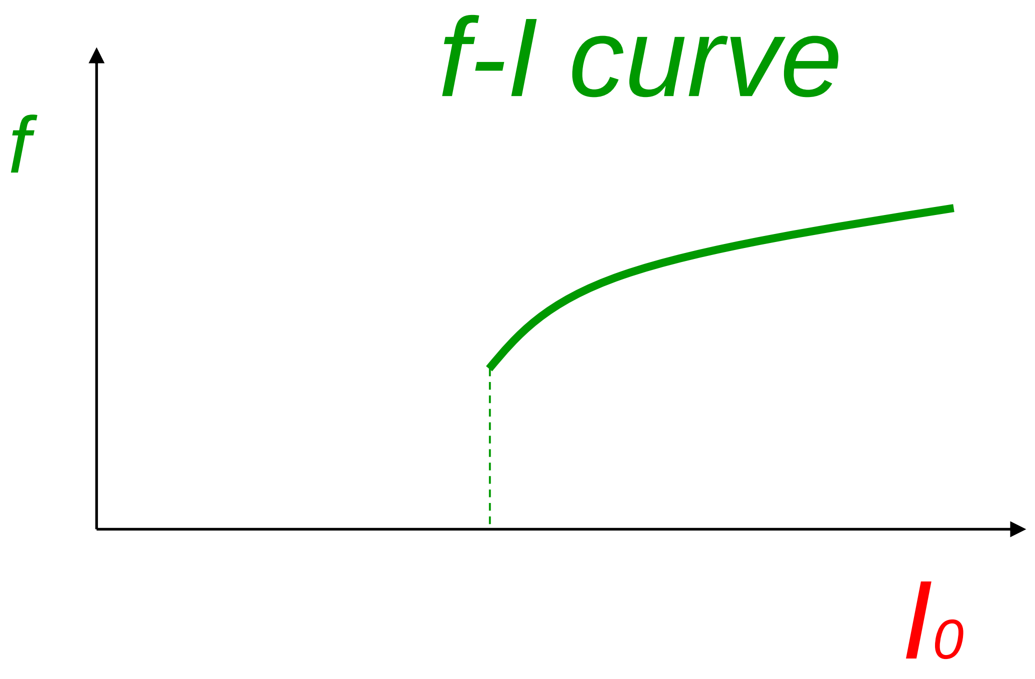


Image:
Neuronal Dynamics, Gerstner et al., Cambridge (2014)

Neuronal Dynamics – Quiz 3.5.

A. Short current pulses. In a 2-dimensional neuron model, the effect of a delta current pulse can be analyzed

[ ] By moving the u-nullcline vertically upward
[ ] By moving the w-nullcline vertically upward
[ ] As a potential change in the stability or number of the fixed point(s)
[ ] As a new initial condition
[ ] By following the flow of arrows in the appropriate phase plane diagram

B. Constant current. In a 2-dimensional neuron model, the effect of a constant current can be analyzed

[ ] By moving the u-nullcline vertically upward
[ ] By moving the w-nullcline vertically upward
[ ] As a potential change in the stability or number of the fixed point(s)
[ ] By following the flow of arrows in the appropriate phase plane diagram

NOW Exercise 2.1: Stability of Fixed Point in 2D

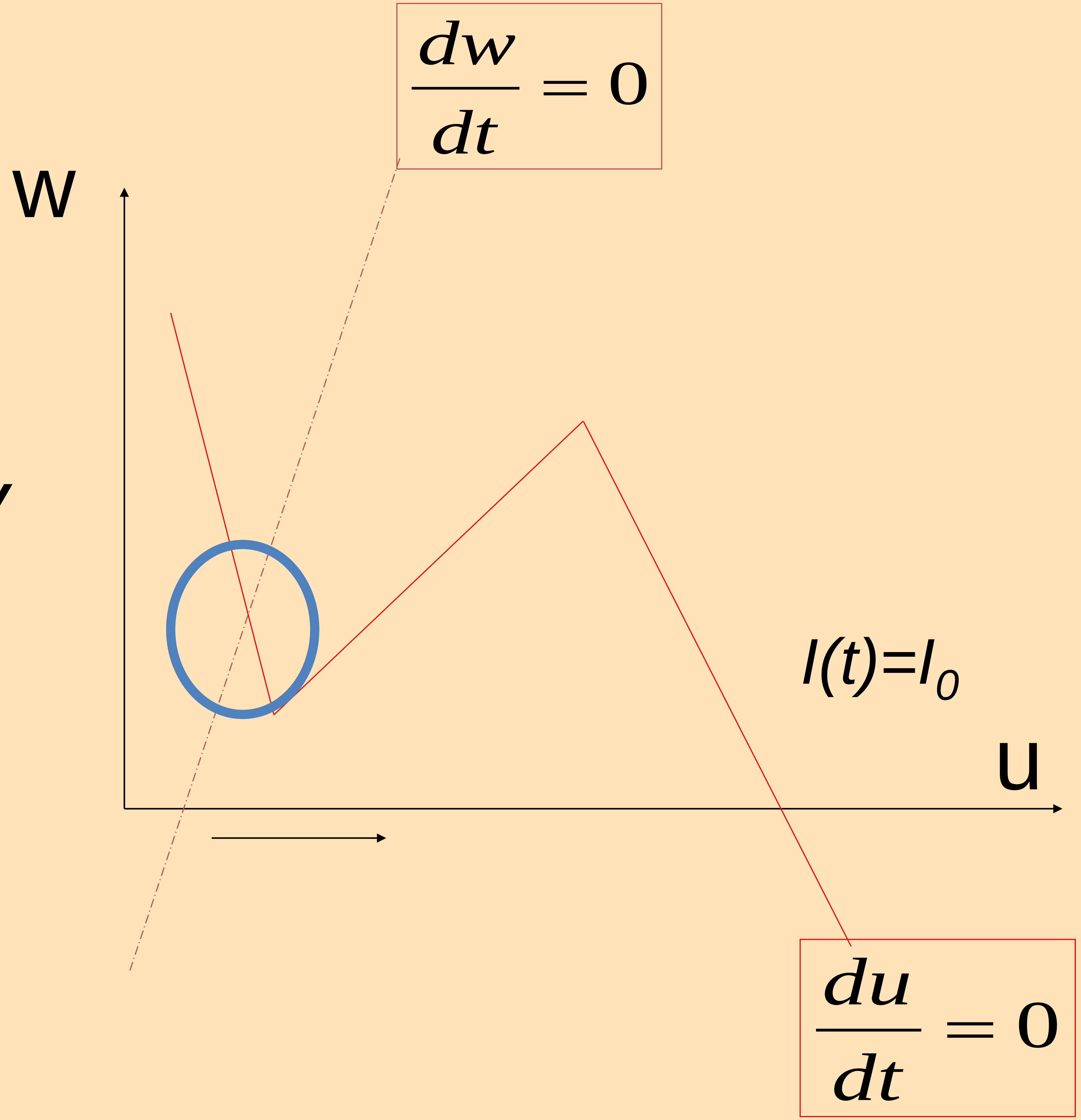
$$\frac{du}{dt} = \alpha u - w$$

$$\frac{dw}{dt} = \beta u - w$$

- calculate *stability*
- compare

$$\frac{dx}{dt} = -\frac{x}{\tau}$$

Exercises:
2.1***now!***
2.2 *homework*



Week 3 – Summary 3.3

Phase plane analysis of neuron models is particularly interesting because the input I only enters into the first variable (voltage u).
As a consequence of this observation, we can discuss two important input scenarios as follows:

1. Constant input. In this case the u-nullcline is shifted vertically.
2. Pulse input. In this case the u-nullcline is not shifted, but the pulse causes a horizontal shift of the initial condition.

Computer exercise now

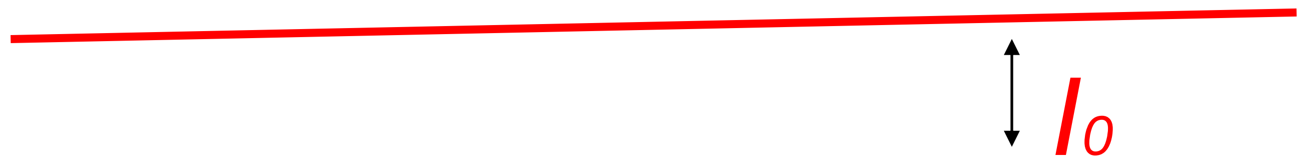
Can we understand the dynamics of the 2D model?

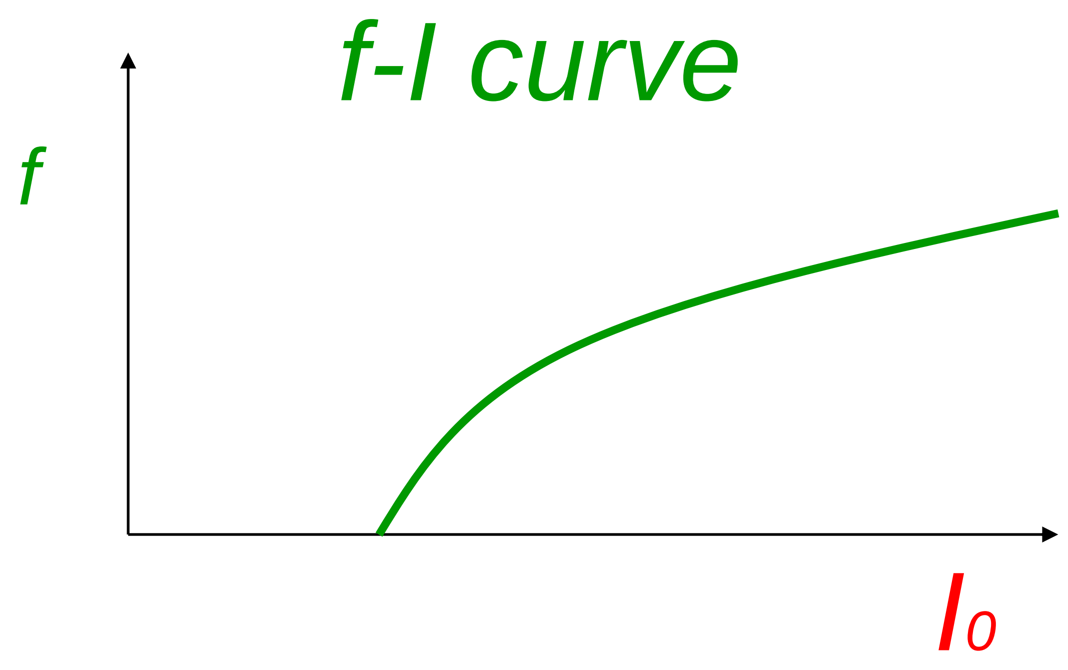
The END for today

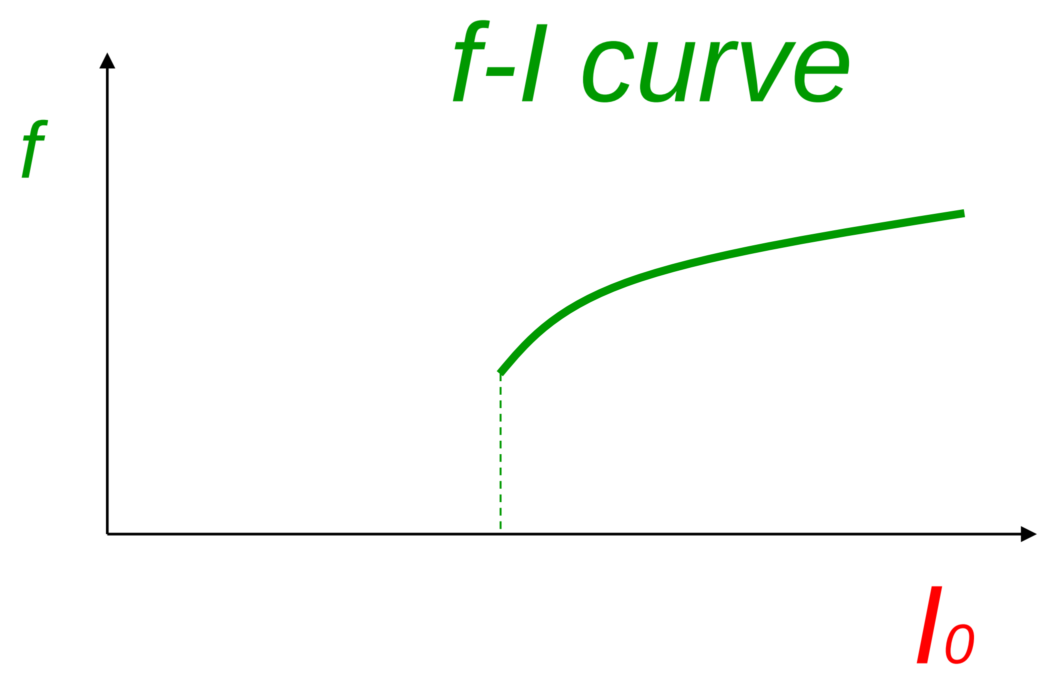
Now: computer exercises

Type I and type II models

ramp input/
constant input







NOW Exercise 2.1: Stability of Fixed Point in 2D

$$\frac{du}{dt} = \alpha u - w$$

$$\frac{dw}{dt} = \beta u - w$$

- calculate *stability*
- compare

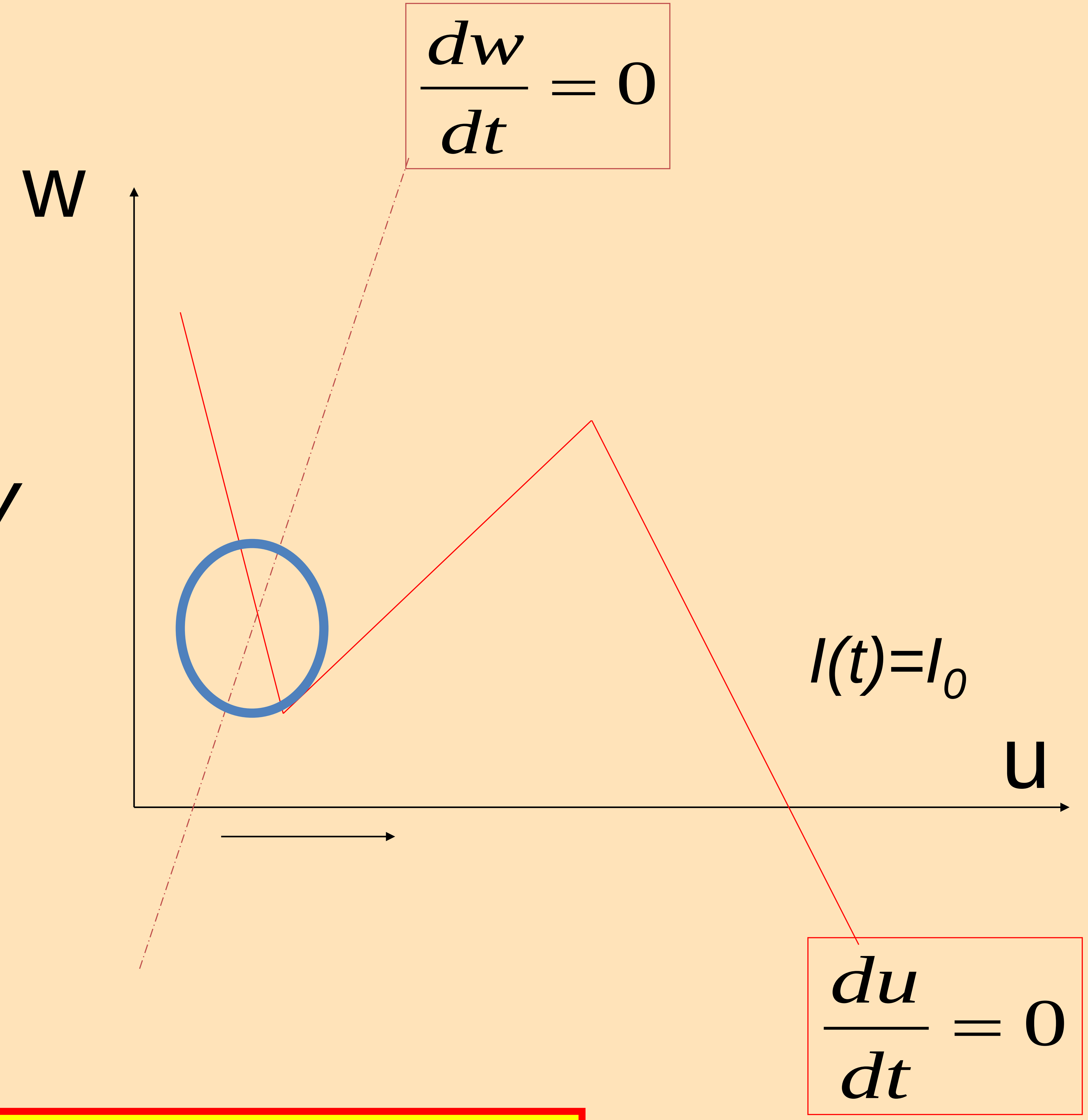
$$\frac{dx}{dt} = -\frac{x}{\tau}$$

Exercises:

2.1 start now!

2.2 *homework*

(you may start if you have time)



Exercise: later

Discussion of Exercise 2 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Fixed point at (u_0, w_0)

At fixed point

$$0 = F(u_0, w_0) + RI_0$$

$$0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

Discussion of Exercise 2 - Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Fixed point at (u_0, w_0)

At fixed point

$$0 = F(u_0, w_0) + RI_0$$

$$0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

$$\tau \frac{dx}{dt} = F_u x + F_w y$$

$$\tau_w \frac{dy}{dt} = G_u x + G_w y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x} \,;$$

Discussion of Exercise 2 Detour. Stability of fixed points

Linear matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x} \,;$$

Search for solution

$$\boldsymbol{x}(t) = \boldsymbol{e}\,\exp(\lambda t)$$

Two solution with Eigenvalues λ_+, λ_-

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

Discussion of Exercise 2: Detour. Stability of fixed points

Linear matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x}$$

Search for solution

$$\boldsymbol{x}(t) = \boldsymbol{e}\,\exp(\lambda t)$$

Two solution with Eigenvalues λ_+, λ_-

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

Stability requires:

$$\lambda_+ < 0 \quad \text{and} \quad \lambda_- < 0$$

$$F_u + G_w < 0$$

and

$$F_u G_w - F_w G_u > 0$$

Discussion of exercise 2: Detour. Stability of fixed points

stimulus

$$\tau \frac{du}{dt} = au - w + I_0$$

$$\tau_w \frac{dw}{dt} = cu - w$$

$$\lambda_{+/-} =$$