Neural Networks and Biological Modeling

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Correction Question Set 8

Exercise 1: Continuous population model

We study the system with lateral connection $w(x-y)$ described by the equation

$$\tau \frac{\partial h(x,t)}{\partial t} = -h(x,t) + \int w(x-y) F[h(y,t)] dy + I_{ext}(x,t) \tag{1}$$

where $F[h(x,t)] = A(x,t)$ is the population activity at point x and at time t.

1.1 With the following conditions,

$$\begin{aligned} I_{ext}(t) &= const. \\ h(x,t) &= h_0 \end{aligned}$$

the equation (1) becomes

$$\begin{aligned} 0 &= -h_0 + I_{ext} + F(h_0) \underbrace{\int w(x-y) dy}_{=\bar{w}} \qquad (2) \\ &= \bar{w} A_0 - h_0 + I_{ext}. \end{aligned}$$

Therefore,

$$A_0 = \frac{h_0 - I_{ext}}{\bar{w}} \tag{3}$$

1.2 Linearizing (1) around h_0, we find

$$\tau \frac{\partial}{\partial t} \Delta h(x,t) = -\Delta h(x,t) + \int w(x-y) F'(h_0) \Delta h(y,t) dy + O(\Delta h^2)$$

where we used (2) to get rid of h_0. Using the following Fourier transform formula for the convolution

$$\left(\int f(x-y) g(y) dy \right)^* = f^*(k) g^*(k)$$

where f^* is the Fourier transform of f, $f^*(k) = \int e^{-ikx} f(x) dx$, we have

$$\tau \frac{\partial}{\partial t} \Delta h^*(k,t) = -\Delta h^*(k,t) + F'(h_0) w^*(k) \Delta h^*(k,t) = (F'(h_0) w^*(k) - 1) \Delta h^*(k,t)\,.$$

Integration once through time,

$$\Delta h^*(k,t) = C(k) e^{-(1-F'(h_0)w^*(k))t/\tau} = C(k) e^{-\kappa(k)t/\tau}\,,$$

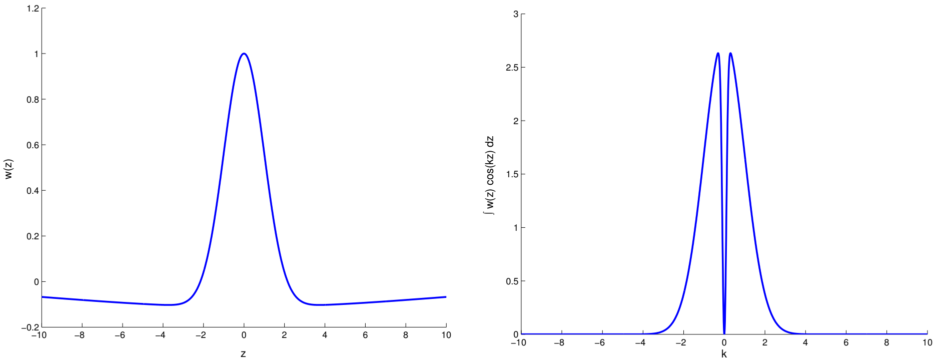
and taking the inverse of the transform,

$$\Delta h(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(k) e^{ikx} e^{-\kappa(k)t/\tau} dk\,.$$

The perturbation evolves as a superposition of modes which has the form $\cos(kx+\varphi(t))e^{-\Re(\kappa(k))t/\tau}$. The stationary state is stable if $\Re(\kappa(k)) > 0$ for all k.

1.3 The function $w(z)$ is shown of figure 1a. It's an excitatory interaction at short distance and an inhibitory at long distance. There is an equilibrium between excitation and inhibition in the sense that $\int_{-\infty}^{+\infty} w(z)dz = 0$. The typical form of this function is often called "mexican hat".

The real part of the Fourier transform, $\int w(z)\cos(kz)dz$ is shown on figure 1b. we see that this function is positive everywhere and its maximal value is about 2.5. From stability condition $\Re(\kappa) > 0$, we deduce that the uniform steady state stability is only standing if the susceptibility $f'(h_0)$ is small enough, i.e. of the order of 0.4. For a better understanding, note that the derivative $f'(h_0)$ represent the variation of the activity due to a small perturbation of the potential: if f' is big, a small perturbation of the potential leads to a big perturbation in the activity of what leads the instability.



(a) Graph of $w(z)$ with $\sigma_1 = 1$ and $\sigma_2 = 10$

(b) Graph of $\int w(z)cos(kz)dz$ with $\sigma_1 = 1$ and $\sigma_2 = 10$.

Figure 1:

Exercise 2: Stationary state in a network with lateral connections

Justification of the assumption that $2d > \sigma$**:** [Proof by contradiction]: Assume $2d < \sigma$. Consider the potential, $h(x)$, at the boundary of the group of active neurons, where $x_1 \lesssim 2d \lesssim x_2$. We find that, if $2d < \sigma$, then, $h(x_1) = h(x_2) = 2d$, but as we suppose that the neuron at x_1 is active and the one at x_2 is not, this leads to $\theta \leq 2d < \theta$ which shows that $2d$ cannot be less than σ.

2.1 The input potential at x_0 is:

$$\begin{aligned} h(x_0) = &\int_{-\infty}^{0} w(x_0,x')A(x')dx' + \int_{0}^{x_0-\sigma} w(x_0,x')A(x')dx' + \int_{x_0-\sigma}^{x_0} w(x_0,x')A(x')dx' \\ &+ \int_{x_0}^{2d} w(x_0,x')A(x')dx' + \int_{2d}^{\infty} w(x_0,x')A(x')dx' \\ &= 0 + (x_0-\sigma)(-b) + \sigma\cdot 1 + (2d - x_0)\cdot 1 + 0 \\ &= (x_0-\sigma)(-b) + \sigma + 2d - x_0 \\ &= \sigma(1+b) - x_0(1+b) + 2d. \end{aligned} \quad (4)$$

2.2 It follows from the definition of $F(h)$:

$$\begin{aligned} h(2d) = \Theta &= (2d - \sigma)(-b) + \sigma + 2d - 2d \\ &= \sigma(1+b) - 2db \end{aligned} \tag{5}$$

We can now calculate d:

$$d = \frac{-\Theta + \sigma(1+b)}{2b} \tag{6}$$

2.3 Neglecting the trivial solution of $d = 0$ the bump's size cannot be infinite (in that case each neuron would receive close to infinite inhibition and it would therefore not be active, which leads to inconsistency). Moreover the $2d$ bump could appear in any location (it is translation-invariant).

Exercise 3: Stability of the stationary bump solution

3.1 Choosing $\Delta t = \tau$ we obtain the time-discretized equation:

$$h(x, t + \Delta t) = \int w(x - y) F[h(y, t)] dy \tag{7}$$

3.2 We consider again a position x_0 close to $x = 2d$ to evaluate the potential:

$$\begin{aligned} h(x_0, t_p + \Delta t) &= \sigma - b(x_0 - \sigma) + 2d + \delta - x_0 \\ &= \underbrace{\sigma(1+b) - x_0(b+1) + 2d}_{=h_0(x_0)} + \delta \end{aligned} \tag{8}$$

3.3 We know from the previous exercise, that the stationary solution fullfills:

$$h_0(2d) = \sigma(1+b) - 2db = \Theta \tag{9}$$

With this we calculate the potential at position $x_0 = 2d + \delta$:

$$\begin{aligned} h(x_0 = 2d + \delta, t_p + \Delta t) &= \underbrace{\sigma(1+b) - 2db}_{=h_0(2d)=\Theta} - b\delta \\ &= \Theta - b\delta < \Theta \end{aligned} \tag{10}$$

Thus, the potential after one time step might have increased a bit due to the pertubation but, for non-zero δ, it is still lower than the threshold. This means that the resulting bump is not a self-consistent solution of the stationary equation investigated in Ex. 2 since the potential at the boundary of the bump isn't equal to the threshold Θ.
If we now update the activity $A(x, t_p + \Delta t) = F[h(x, t_p + \Delta t)]$ we get a smaller (i.e. smaller width D) bump than the perturbed one ($A(x, t_p)$). During the following time steps the same thing will happen; but we start with an activity bump which is still perturbed but less than the original pertubation: $2d < D < 2d + \delta$. Following the procedure above the bump size will decrease further and further until it reaches the stationary self-consistent state $D = 2d$, which proofs its stability.

3.4 At time $t_p + \Delta t$ we find the new bump size $D = 2d + \delta(t_p + \Delta t)$ from the intersection point of $h(x, t_p + \Delta t)$ with the threshold Θ. Again exploiting

$$h_0(2d) = \sigma(1+b) - 2db = \Theta \tag{11}$$

we find

$$\begin{aligned} h(x, t_p + \Delta t) &= -x(b+1) + \sigma(1+b) + 2d + \delta(t_p) \\ h(x, t_p + \Delta t) &= -x(b+1) + \Theta + 2db + 2d + \delta(t_p). \end{aligned} \tag{12}$$

The intersection $h(x = 2d + \delta(t_p + \Delta t), t_p + \Delta t) = \Theta$ thus leads to:

$$\begin{aligned} \Theta &= -(2d + \delta(t_p + \Delta t))(b+1) + \Theta + 2d + \delta(t_p) + 2db \\ (b+1)(2d + \delta(t_p + \Delta t)) &= 2d + \delta(t_p) + 2db \\ \delta(t_p + \Delta t) &= \frac{1}{b+1}\delta(t_p) < \delta(t_p). \end{aligned} \tag{13}$$

Accordingly, the pertubation is smaller after one time step. As above, or e.g. by complete induction, one can argue that this is also true for all following time steps and we get

$$\delta(t_p + n \cdot \Delta t) = \left(\frac{1}{b+1}\right)^n \delta(t_p) \quad \text{for} \quad n \in \mathbb{N}_+ \tag{14}$$

Thus the pertubation δ decays to zero for $n \to \infty$ as expected for a stable solution. The functional form of the decay is an exponential.