SOLUTION SUGGESTIONS SÉRIE 9 - EVEN NUMBERED EXERCISES

Solution (Exercise 2)**. Hint for 1 and 2** Let $\{f_1, f_2, ..., f_n\}$ be an arbitrary orthonormal basis of $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map s.t. $T(e_i) = f_i$ where $\{e_1, ..., e_n\}$ is the standard orthonormal basis. Since T maps the standard orthonormal basis to an orthonormal basis, it is orthogonal, i.e. the matrix of T in the standard basis satisfies $M_T^t = M_T^{-1}$.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix M_ϕ in the standard basis. Its matrix in the basis $\{f_1, ..., f_n\}$ is given by $M_T M_\phi M_T^{-1} = M_T M_\phi M_T^t$. From this the reader can easily check that if the matrix of ϕ is orthogonal in one orthonormal basis the same is true w.r.t. any orthonormal basis.

Hint for 4

From the expression for the matrix in different bases given above, we see that the determinant of a matrix is independent of the choice of basis and is therefore attached to the linear transformation.

Hint for 3

Let $\{v_1, ..., v_n\}$ be an arbitrary basis of $\mathbb{R}^n$. We know that the matrix of ϕ is orthogonal w.r.t. this basis iff $< \phi(v_i), \phi(v_j) >= \delta_{i=j}$. Further $< \phi(v_i), \phi(v_j) >=< v_i, v_j >$, therefore the matrix is orthogonal w.r.t this basis iff the basis is orthonormal.

Solution (Exercise 4)**.** This exercise deals with the action of orthogonal maps – $Isom(\mathbb{R}^n)_0$ on the set of orthonormal bases of $\mathbb{R}^n$ – $\mathcal{BO}$.

(1) Let $\{e_1, e_2, e_3, ..., e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Let $\{f_1, f_2, ..., f_n\}$ be another orthonormal basis. Let the linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\phi(e_i) = f_i$, the transformation ϕ is orthogonal since it maps the standard orthonormal basis to another orthonormal basis. This construction shows that the action of orthogonal maps on $\mathcal{BO}$ is transitive.

An orthogonal map fixing the standard orthonormal basis is identity, since orthogonal maps are linear. This shows that the stabiliser of any element of $\mathcal{BO}$ is trivial. So the map $\phi \in Isom(\mathbb{R}^n)_0 \mapsto \{\phi(e_1), ..., \phi(e_n)\} \in \mathcal{BO}$ induces a bijection

$$Isom(\mathbb{R}^n)_0 \simeq \mathcal{BO}$$

(2) Given a vector $e \in S^{n-1}$ we can complete it to an element of $\mathcal{BO}$ i.e. $\exists\ f_2, ..., f_n$ s.t. $\{e, f_2, ..., f_n\} \in \mathcal{BO}$. This follows since any set of linearly independent vectors can be completed to a basis and applying Gram-Schmidt to make the basis orthonormal. Transitivity of action of $Isom(\mathbb{R}^n)_0$ on S^{n-1} follows easily from the transitivity of action of $Isom(\mathbb{R}^n)_0$ on $\mathcal{BO}$.

(3) Let $e \in S^{n-1}$ be any vector, by part (2) $\exists\ \psi \in Isom(\mathbb{R}^n)_0$ s.t. $\psi(e_n) = e$. You can check that

$$Isom(\mathbb{R}^n)_{0,e} = \psi Isom(\mathbb{R}^n)_{0,e_n}\psi^{-1}$$

Therefore

$$Isom(\mathbb{R}^n)_{0,e} \simeq Isom(\mathbb{R}^n)_{0,e_n}$$

(4) Let $\phi \in Isom(\mathbb{R}^3)_{0,e_3}$ have basis M_ϕ w.r.t. the standard orthonormal basis. Suppose

$$M_\phi = \begin{pmatrix} a & b & u \\ c & d & v \\ x & y & z \end{pmatrix}$$

we have $\phi(e_3) = ue_1 + ve_2 + ze_3$, it follows that $u = 0, v = 0, z = 1$. $<\phi(e_1), \phi(e_3)> = <e_1, e_3> = 0$, on the other hand $<\phi(e_1), \phi(e_3)> = <\phi(e_1), e_3> = x$. Similarly, $<\phi(e_2), \phi(e_3)> = <e_2, e_3> = 0$, on the other hand $<\phi(e_2), \phi(e_3)> = <\phi(e_2), e_3> = y$. Further

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the result follows.

(5) Let $\phi \in Isom(\mathbb{R}^n)_{0,e_n}$ have basis $M_\phi = (f_{ij})_{1\leq i,j\leq n}$ w.r.t. the standard orthonormal basis. Here $f_{ij} = <\phi(e_j), e_i>$. Firstly $f_{in} = <\phi(e_n), e_i> = <e_n, e_i> = \delta_{i=n)}$. Further $f_{ni} = <\phi(e_i), e_n> = <\phi(e_i), \phi(e_n)> = <e_i, e_n> = \delta_{i=n}$. Observe that the upper block $\widetilde{M_\phi} = (f_{ij})_{1\leq i,j\leq n-1}$ satisfies $\widetilde{M_\phi}^t \widetilde{M_\phi} = \widetilde{M_\phi}\widetilde{M_\phi}^t = Id_{n-1\times n-1}$ so is orthogonal.

Solution (Exercise 6)**.** Let $\phi_{V,W} : V \to W$ be a surjective linear isometry. If we have $\phi_{V,W}(v) = 0$ for some $v \in V$ i.e. $|\phi_{V,W}(v) - 0| = 0$, we conclude that $|v - 0| = 0$ since $\phi_{V,W}$ is a linear isometry. So $\phi_{V,W}$ is also injective and hence $dim(V) = dim(W)$.

Let us denote $dim(V) = m$. Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ s.t. $v_1, ..., v_m \in V$ (and so form an orthonormal basis of

V). Note this is possible since starting with an orthonormal basis of V extend it to a basis of $\mathbb{R}^n$ and apply Gram-Schmidt procedure to get a basis with the given property. Similarly $\{w_1, w_2, ..., w_n\}$ be an orthonormal basis of $\mathbb{R}^n$ s.t. $w_1, ..., w_m \in W$. Define a linear transformation $\phi : V \to W$ by $\phi(v_i) = \phi_{V,W}(v_i)$ for $1 \leq i \leq m$ and $\phi(v_i) = w_i$ for $m+1 \leq i \leq n$.

Observe that $\phi(v) = \phi_{V,W}(v)$ for all $v \in V$. Further $\{\phi(v_1), ..., \phi(v_n)\}$ are orthonormal (since $\phi_{V,W}$ is a linear isometry into W and $\{w_{m+1}, ..., w_n\}$ form an orthonormal basis of $W^\perp$) So ϕ is a linear isometry of $\mathbb{R}^n$ extending $\phi_{V,W}$ as desired.

Solution (Exercise 8)**.** We proceed as in Ex 1 to determine if the matrix is orthogonal and if so we determine its nature by calculating the eigenvalues. We can find the type just by looking at the trace and determinant:

$$\frac{1}{9}\begin{pmatrix} 8 & 1 & 4 \\ 1 & 8 & -4 \\ 4 & -4 & -7 \end{pmatrix}$$

is orthogonal and is a reflection about the plane $\mathbb{R}v_1 \bigoplus \mathbb{R}v_2$ where $v_1 = (1,1,0)$ and $v_2 = (4,0,1)$ in the standard basis.

$$\frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & -4 \\ 4 & -4 & -7 \end{pmatrix}$$

is not orthogonal

$$\frac{1}{3}\begin{pmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{pmatrix}$$

is orthogonal and its type is a rotation with axis $\mathbb{R}(0,2,1)$ and angle $\pm\arccos(\frac{2}{3})$.

$$\frac{1}{25}\begin{pmatrix} -9 & -12 & -20 \\ -20 & 15 & 0 \\ -12 & -16 & 15 \end{pmatrix}$$

is orthogonal and its type is an antirotation with axis $\mathbb{R}(2,1,1)$ and angle $\pm\arccos(\frac{23}{25})$.

Solution (Exercise 10)**.** Let

$$\varphi(x,y,z) = (X,Y,Z)$$

with

$$X = \frac{1}{9}(x - 8y + 4z) - 1$$

$$Y = \frac{1}{9}(4x + 4y + 7z) + 2$$
$$Z = \frac{1}{9}(-8x + y + 4z) + 2$$

(1) First we look at the linear part:

$$\frac{1}{9}\begin{pmatrix} 1 & -8 & 4 \\ 4 & 4 & 7 \\ -8 & 1 & 4 \end{pmatrix}$$

This is a rotation with axis $\mathbb{R}(-1, 2, 2)$ and angle $\frac{\pi}{2}$. The translation vector is parallel to the axis so this is a vissage. (φ has no fixed points.)

(2) let $\psi(x, y, z) = (X', Y', Z')$ with

$$X' = \frac{1}{3}(x + 2y + 2z) + 1$$
$$Y' = \frac{1}{3}(2x + y - 2z) - 1$$
$$Z' = \frac{1}{3}(2x - 2y + z) - 1.$$

First we look at the linear part:

$$\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

This is a reflection about the plane $\mathbb{R}(1, 1, 0) \bigoplus \mathbb{R}(1, 0, 1) = (\mathbb{R}(-1, 1, 1))^{\perp}$. The translation vector is perpendicular to the plane. The set of fixed points is the affine plane $(3/2, 0, 0) + \mathbb{R}(1, 1, 0) \bigoplus \mathbb{R}(1, 0, 1)$, ψ is just the reflection w.r.t. this plane.

(3) The nature of $\varphi \circ \psi \circ \varphi^{-1}$: By the proof of exercise 9.1, the transformation $\varphi \circ \psi \circ \varphi^{-1}$ is a reflection about the affine plane $\varphi(3/2, 0, 0) + \mathbb{R}\varphi(1, 1, 0) \bigoplus \mathbb{R}\varphi(1, 0, 1)$.

□

Solution (Exercise 12)**.** Let $a, b, c, d, e, f \in \mathbb{R}$. Consider $\varphi(x, y, z) = (X, Y, Z)$ in the standard basis with

$$\begin{aligned} X &= \frac{1}{d}(2x - 2y + az) + 1 \\ Y &= \frac{1}{d}(x + by + 2z) + e \\ Z &= \frac{1}{d}(cx - y + 2z) + f \end{aligned}$$

(1) First let us look at the linear part:

$$\frac{1}{d}\begin{pmatrix} 2 & -2 & a \\ 1 & b & 2 \\ c & -1 & 2 \end{pmatrix}$$

Using the fact that the linear part is an orthogonal matrix the rows are orthogonal to each other, we get

$$a = 1, b = 2, c = -2$$

and using the fact that each row has norm 1 we get

$$d = \pm 3$$

For φ to be a vissage, the linear part has to be special orthogonal, we conclude that

$$d = 3$$

. The reader may calculate that the axis of this rotation is $\mathbb{R}(-1, 1, 1)$. Further for φ to be a vissage, the translation vector is not in $\mathrm{Im}(\varphi_0 - \mathrm{id})$. i.e.

$$< (-1, 1, 1), (1, e, f) > \neq 0 \iff e + f \neq 1$$

(2) For φ to be an anti rotation, from calculations as above we have

$$(a, b, c, d) = (1, 2, -2, -3)$$

and e, f are arbitrary reals.

(3) Let us assume that φ is not a vissage. From part (1) and part (2), we know that the linear part is either

$$\frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix}$$

or

$$\frac{-1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix}$$

In the first case φ_0 is a rotation with axis $\mathbb{R}(-1, 1, 1)$ and angle $\pm\frac{\pi}{3}$ and in the second case φ_0 is an anti rotation with axis $\mathbb{R}(-1, 1, 1)$ and angle $\pm\frac{\pi}{3}$. In both these cases,

$$\varphi_0^6 = \mathrm{Id}_{\mathbb{R}^3}$$

. Further since φ is not a vissage we have $\varphi(v) = \varphi_0(v) + u$ with $u = (\varphi_0 - \mathrm{Id})w$ for some $w \in \mathbb{R}^3$. We get

$$\varphi(v) = \varphi_0(v + w) - w$$

and inductively you can see that

$$\varphi^k(v) = \varphi_0^k(v+w) - w$$

In particular

$$\varphi^6(v) = \varphi_0^6(v+w) - w = v + w - w = v$$

since $\varphi_0^6 = \mathrm{Id}_{\mathbb{R}^3}$.

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