MCAA lecture 1

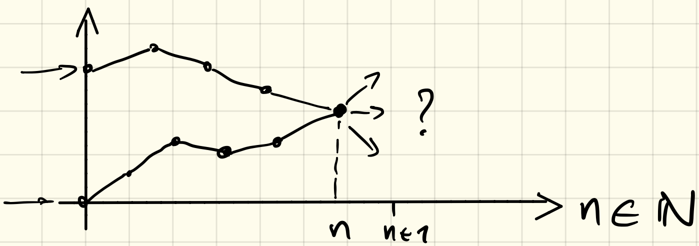
Definition: A time-homogeneous Markov chain is a discrete-time stochastic process $(X_n, n \in \mathbb{N})$ with values in a finite or countable set S (= the state space) such that:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= \mathbb{P}(X_{n+1} = j \mid X_n = i) = p_{ij} \qquad \forall n \in \mathbb{N},\ j, i, i_{n-1} \ldots i_0 \in S$$

Markov property

time homogeneous

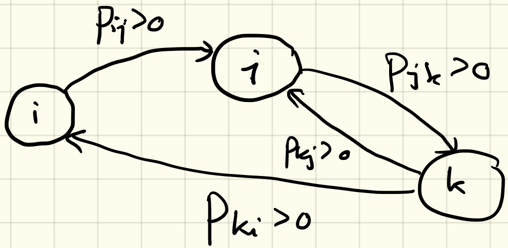


Transition matrix: $P = (p_{ij})_{i,j \in S}$

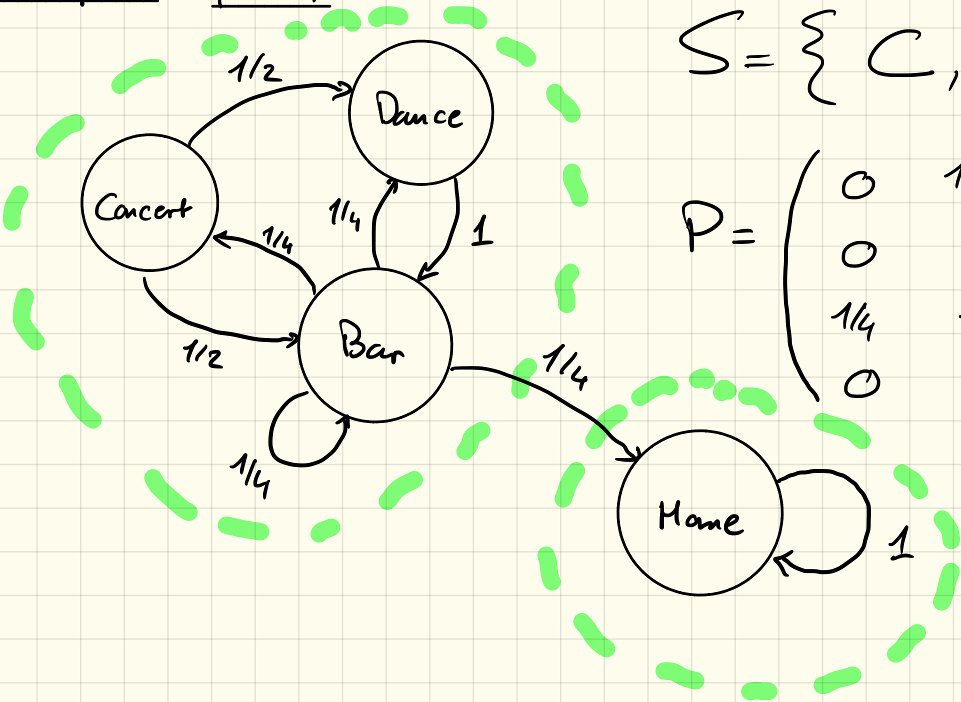
- $0 \leq p_{ij} \leq 1 \quad \forall i,j \in S$
- $\sum_{j \in S} p_{ij} = \sum_{j \in S} \mathbb{P}(X_{n+1} = j \mid X_n = i) = 1 \quad \forall i \in S$

  $\left(= \mathbb{P}(X_{n+1} \in S \mid X_n = i) \right)$
- $\sum_{i \in S} p_{ij} = \sum_{i \in S} \mathbb{P}(X_{n+1} = j \mid X_n = i) \in [0, +\infty]$

Transition graph:



Example 1: party



$$S = \{ C, D, B, H \}$$

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 2: Simple symmetric random walk

State space: $S = \mathbb{Z}$

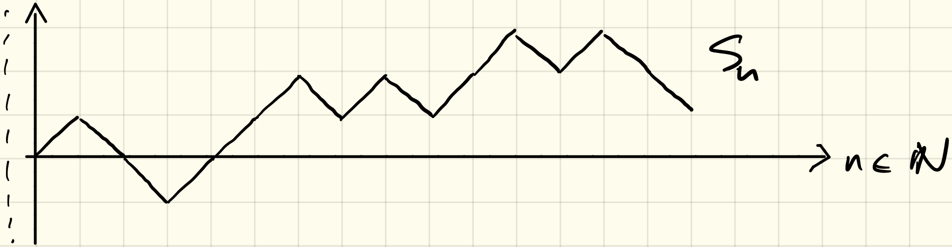
Let $(X_n, n \geq 1)$ be iid random variables

[i.e. independent & identically distributed]

such that $\mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = \frac{1}{2} \quad \forall n \geq 1$

$$S_0 = 0, \; S_n = X_1 + \dots + X_n \quad n \geq 1$$

Claim: The process $(S_n, n \in \mathbb{N})$ is a time-homogeneous Markov chain.



$$P = (p_{ij})_{i,j \in \mathbb{Z}}$$

$$p_{ij} = \begin{cases} 1/2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof:

$$\mathbb{P}(S_{n+1} = j \mid S_n = i, S_{n-1} = i_{n-1} \dots S_0 = i_0)$$

$$= \mathbb{P}(S_n + X_{n+1} = j \mid S_n = i, S_{n-1} = i_{n-1} \dots S_0 = i_0)$$

$$= \mathbb{P}(\underbrace{X_{n+1} = j-i}_{} \mid \underbrace{S_n = i, S_{n-1} = i_{n-1} \dots S_0 = i_0}_{})$$

independent

$$= \mathbb{P}(X_{n+1} = j-i) = \begin{cases} \frac{1}{2} & \text{if } |j-i| = 1 \\ 0 & \text{otherwise} \end{cases}$$

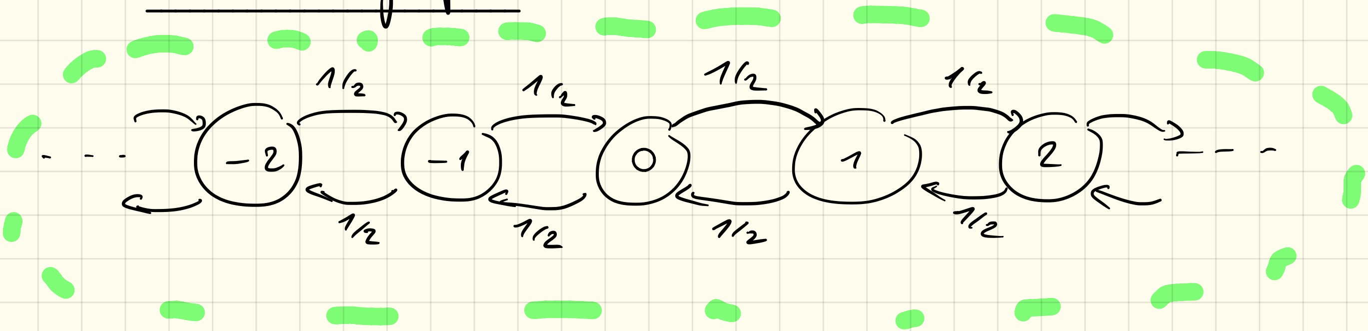
$$\mathbb{P}(S_{n+1} = j \mid S_n = i) = \underset{\text{(as above)}}{\dots\dots} = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Transition matrix:

$$p_{ij} = \begin{cases} 1/2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & \frac{1}{2} & & 0 \\ & \frac{1}{2} & \ddots & \ddots & \\ & & \ddots & 0 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & \ddots \end{pmatrix}$$

Transition graph:



Distribution of the Markov chain at time n:

$$\pi_i^{(n)} = \mathbb{P}(X_n = i) \qquad n \in \mathbb{N},\ i \in S$$

Initial distribution: $\pi_i^{(0)} = \mathbb{P}(X_0 = i) \qquad i \in S$

$$\left[\ 0 \le \pi_i^{(n)} \le 1,\quad \sum_{i \in S} \pi_i^{(n)} = 1 \qquad \forall n \in \mathbb{N}\ \right]$$

$$\pi_j^{(n+1)} = \mathbb{P}(X_{n+1} = j) = \sum_{i \in S} \mathbb{P}(X_{n+1} = j, X_n = i)$$

$$= \sum_{i \in S} \underbrace{\mathbb{P}(X_{n+1} = j \mid X_n = i)}_{= p_{ij}} \underbrace{\mathbb{P}(X_n = i)}_{\pi_i^{(n)}} = \sum_{i \in S} \pi_i^{(n)} p_{ij}$$

In vector form: $\pi^{(n+1)} = \pi^{(n)} \cdot P \implies \pi^{(n)} = \pi^{(0)} \cdot P^n$

Questions (for the 1st part of the class)

A. When does $\pi^{(n)}$ converge as $n \to \infty$ towards a limiting distribution π ?

B. When it converges, at what rate does it converge (i.e. is $\pi^{(n)}$ any close to π for a given value n) ?

m-step transition probabilities:

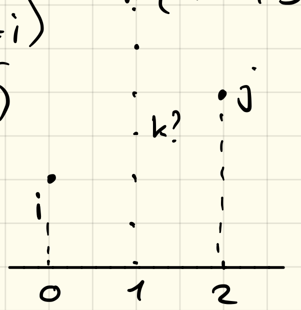
$$p_{ij}^{(m)} = \mathbb{P}(X_{n+m}=j \mid X_n = i) \underset{\uparrow}{=} \mathbb{P}(X_m = j \mid X_0 = i)$$

time-homogeneity

Chapman-Kolmogorov equations:

$m=2$: $$p_{ij}^{(2)} = \mathbb{P}(X_2 = j \mid X_0 = i) = \sum_{k\in S} \underbrace{\mathbb{P}(X_2 = j, X_1 = k \mid X_0 = i)}_{\mathbb{P}(A\cap B \mid C)}$$

$$= \sum_{k\in S} \underbrace{\mathbb{P}(X_2 = j \mid X_1 = k, \overset{\text{Markov}}{\cancel{X_0 = i}})}_{\mathbb{P}(A \mid B\cap C)} \cdot \underbrace{\mathbb{P}(X_1 = k \mid X_0 = i)}_{\mathbb{P}(B\mid C)}$$



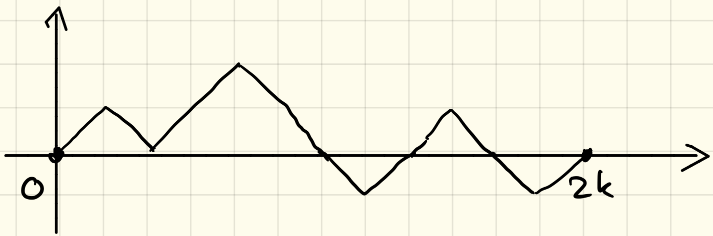
$$= \sum_{k\in S} p_{kj} \cdot p_{ik} = \sum_{k\in S} p_{ik} \cdot p_{kj}$$

$$= (P\cdot P)_{ij} = (P^2)_{ij}$$

$$p_{ij}^{(m)} = (P^m)_{ij}$$

Example: simple symmetric random walk

$$p_{00}^{(m)} = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \binom{2k}{k} \cdot \frac{1}{2^{2k}} & \text{if } m \text{ is even } (=2k) \end{cases}$$



Classification of states

Definitions

- Two states $i, j \in S$ communicate ("$i \leftrightarrow j$")
  if $\exists\, n, m \geq 0$ such that $p_{ij}^{(n)} > 0$ & $p_{ji}^{(m)} > 0$
  [convention: $p_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$; so $i \leftrightarrow i \ \ \forall i \in S$]

- The relation $i \leftrightarrow j$ is:
  - reflexive ($i \leftrightarrow i \ \ \forall i \in S$)
  - symmetric ($i \leftrightarrow j$ iff $j \leftrightarrow i \ \ \forall i, j \in S$)
  - transitive (if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k \ \ \forall i, j, k \in S$)

Proof of the transivity:

if $\exists n \geq 0$ st $p_{ij}^{(n)} > 0$ & $\exists m \geq 0$ s.t. $p_{jk}^{(m)} > 0$

then $p_{ik}^{(n+m)} = \left(P^{n+m}\right)_{ik} = \sum_{\ell \in S} \left(P^{n}\right)_{i\ell} \left(P^{m}\right)_{\ell k}$

$$= \sum_{\ell \in S} p_{i\ell}^{(n)} p_{\ell k}^{(m)} \geq p_{ij}^{(n)} \cdot p_{jk}^{(m)} > 0 \quad \#$$

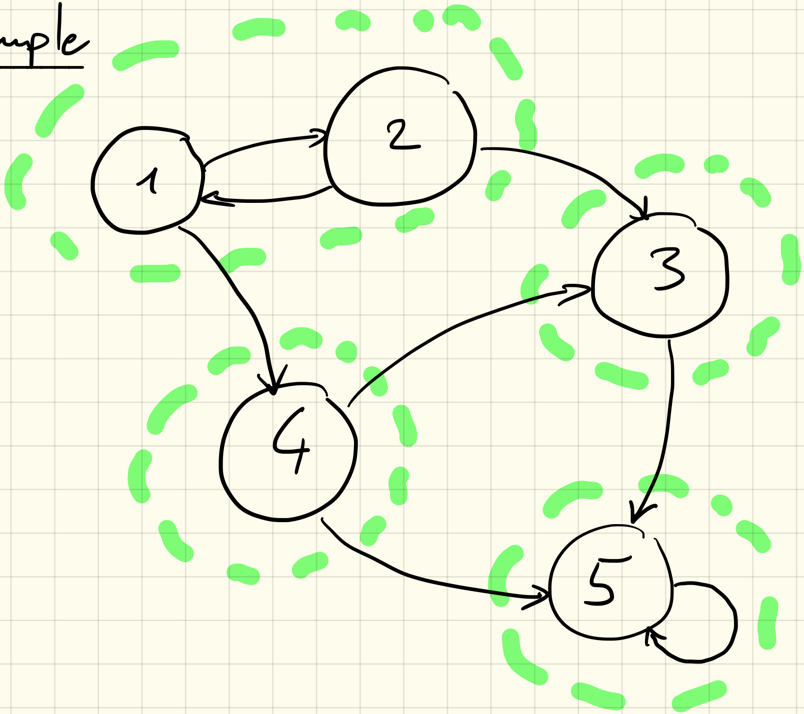
So the relation $i \leftrightarrow j$ is an <u>equivalance relation</u>.

The state space S can therefore be partitioned into <u>equivalence classes</u> ($i \leftrightarrow j$ iff i,j belong to the same class).

<u>Defs</u>: • A Markov chain is <u>irreducible</u> if all states communicate (only one class)

• A state i is <u>absorbing</u> if $p_{ii} = 1$

Example



Periodicity

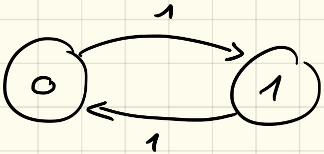
Def: For a state $i \in S$, define $d_i = \gcd(n \geq 1 : p_{ii}^{(n)} > 0)$

- If $d_i = 1$, we say that state i is aperiodic
- If $d_i > 1$, we say that state i is periodic with period d_i.

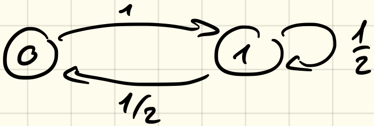
Facts:

- In a given equivalence class, all states have the same period $d_i \equiv d$.
- In there is at least one self-loop in the class, then all states are aperiodic in this class.

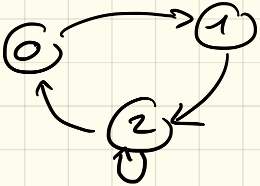
Examples:

- 

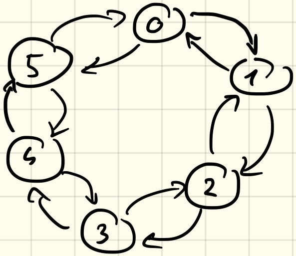
$d = 2$

- 

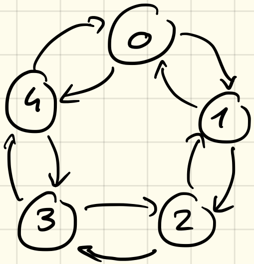
$d = 1$

- 

$d = 1$

- 

$d = 2$

- 

$d = 1$