MCAA lecture 2

Recurrence & transience

Let $(X_n, n \geq 1)$ be a Markov chain with state space S, initial distribution $\pi^{(0)}$ & transition matrix P.

For $i \in S$, define $f_{ii} = \mathbb{P}(\exists n \geq 1 \text{ such that } X_n = i \mid X_0 = i) \in [0,1]$

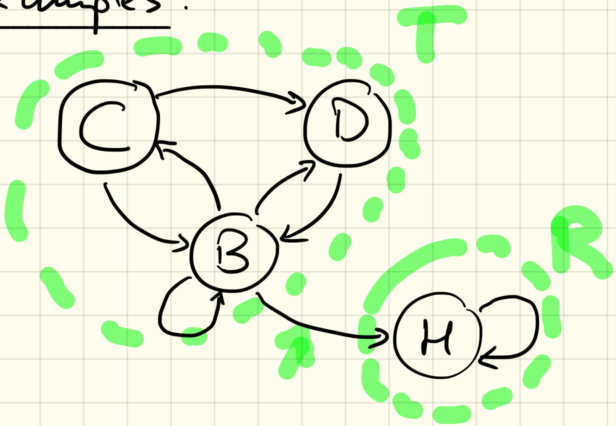
$$= \mathbb{P}\Big(\bigcup_{n \geq 1} \{X_n = i\} \mid X_0 = i \Big)$$

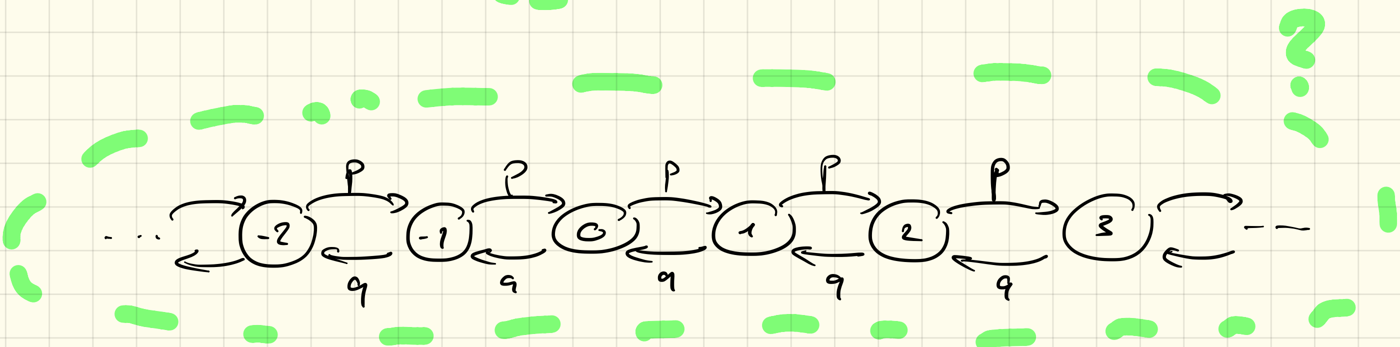
<u>Definition</u>: • A state $i \in S$ is <u>recurrent</u> if $f_{ii} = 1$

• A state $i \in S$ is <u>transient</u> if $f_{ii} < 1$

<u>Remark</u>: recurrent does <u>not</u> mean "$\exists n \geq 1$ s.t. $\mathbb{P}(X_n = i \mid X_0 = i) = 1$"

Examples:





$$p+q=1$$

$$0<p,q<1$$

Facts:

- In a given equivalence class, either all states are recurrent, or all states are transient.

- In a finite chain (i.e. with S finite), an equivalence class is recurrent iff there is no arrow leading out of it. In particular, a finite & irreducible chain is recurrent.

- In an infinite chain, things are more complicated.

Recurrence & transience of infinite chains

Preliminary step:

Define: $p_{ii}^{(n)} \equiv p_{ii}(n) = \mathbb{P}(X_n = i \mid X_0 = i)$ [convention: $p_{ii}(0) = 1$]

$$f_{ii}^{(n)} \equiv f_{ii}(n) = \mathbb{P}(\underbrace{X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i}_{\text{first return time to state } i \,=\, n} \mid X_0 = i)$$

[convention: $f_{ii}(0) = 0$]

Lemma: $\forall n \geq 1$, $\forall i \in S$, we have:

$$p_{ii}(n) = \sum_{m=1}^{n} f_{ii}(m) \cdot p_{ii}(n-m)$$

Proof: Let $\begin{cases} A_n = \{X_n = i\} \\ B_n = \{X_n = i, X_{n-1} \neq i \dots X_1 \neq i\} \end{cases}$ $\quad p_{ii}(n) = \mathbb{P}(A_n \mid X_0 = i)$, $\quad f_{ii}(n) = \mathbb{P}(B_n \mid X_0 = i)$

Observation: if A_n happens, then it must be the case that one of the events $B_1 \dots B_n$ also happens.

i.e. $A_n \subset \bigcup_{m=1}^{n} B_m$

$$p_{ii}(n) = \mathbb{P}(A_n \mid X_0 = i) = \mathbb{P}\left(A_n \cap \left(\bigcup_{m=1}^{n} B_m\right) \mid X_0 = i\right)$$

$$= \sum_{m=1}^{n} \mathbb{P}(A_n \cap B_m \mid X_0 = i) = \sum_{m=1}^{n} \mathbb{P}(A_n \mid B_m, X_0 = i) \cdot \mathbb{P}(B_m \mid X_0 = i)$$

$$= \sum_{m=1}^{n} \underbrace{\mathbb{P}(X_n = i \mid X_m = i}_{p_{ii}(n-m)}, X_{m-1} \neq i, \dots, X_1 \neq i, X_0 = i) \cdot \underbrace{\mathbb{P}(X_m = i, X_{m-1} \neq i \dots X_1 \neq i \mid X_0 = i)}_{= f_{ii}(m)}$$

Markov

#

Proposition

A state $i \in S$ is recurrent (i.e. $f_{ii} = 1$)
iff $\sum_{n \geq 0} p_{ii}(n) = +\infty$

So: a state $i \in S$ is transient (i.e. $f_{ii} < 1$) iff $\sum_{n \geq 0} p_{ii}(n) < +\infty$

Proof:

$$f_{ii} = \mathbb{P}(\exists n \geq 1 \text{ s.t. } X_n = i \mid X_0 = i) = \mathbb{P}\Big(\underbrace{\bigcup_{n\geq 1} A_n}\ \Big| X_0 = i\Big)$$

$$= \mathbb{P}(\exists n \geq 1 \text{ s.t. } X_n = i \text{ for the first time} \mid X_0 = i)$$

$$= \mathbb{P}\Big(\underbrace{\bigcup_{n \geq 1} B_n}\ \Big| X_0 = i\Big) = \sum_{n \geq 1} \underbrace{\mathbb{P}(B_n \mid X_0 = i)}_{= f_{ii}(n)}$$

To be proven: $\sum_{n \geq 0} f_{ii}(n) = +1$ iff $\sum_{n \geq 0} p_{ii}(n) = +\infty$

Lemma: $p_{ii}(n) = \sum_{m=1}^{n} f_{ii}(m) \cdot p_{ii}(n-m) \quad \forall n \geq 1$

= convolution relation $\longrightarrow$ use generating functions!

Define:
$$\begin{cases} P_{ii}(s) = \sum_{n \geq 0} s^n p_{ii}(n) & s \in [0,1] \\ F_{ii}(s) = \sum_{n \geq 0} s^n f_{ii}(n) & s \in [0,1] \end{cases}$$

Fact: (Abel's thm)

Let $(a_n, n \geq 0)$ be a sequence of numbers s.t. $0 \leq a_n \leq 1$ $\forall n$

Then $A(s) = \sum_{n \geq 0} s^n \cdot a_n$ converges $\forall |s| < 1$

and
$$\begin{cases} \text{either } \lim_{s \uparrow 1} A(s) = \sum_{n \geq 0} a_n \in \mathbb{R}_+ \\ \text{or both } \lim_{s \uparrow 1} A(s) = +\infty \text{ and } \sum_{n \geq 0} a_n = +\infty \end{cases}$$

For $0 \leq s < 1$, we have:

$$P_{ii}(s) = 1 + \sum_{n \geq 1} s^n \cdot p_{ii}(n) \overset{\text{lemma}}{=} 1 + \sum_{n \geq 1} s^n \left(\sum_{m=1}^{n} f_{ii}(m) \cdot p_{ii}(n-m) \right)$$

$$= 1 + \underbrace{\sum_{n \geq 1} \sum_{m=1}^{n}} s^m \cdot s^{n-m} f_{ii}(m)\, p_{ii}(n-m)$$

$$= 1 + \sum_{m \geq 1} \sum_{n \geq m} s^m \cdot s^{n-m} f_{ii}(m)\, p_{ii}(n-m)$$

$$= 1 + \underbrace{\sum_{m \geq 1} s^m f_{ii}(m)}_{= F_{ii}(s)} \cdot \underbrace{\sum_{n \geq m} s^{n-m} p_{ii}(n-m)}_{= \sum_{k \geq 0} s^k p_{ii}(k) = P_{ii}(s)}$$

i.e. $P_{ii}(s) = 1 + F_{ii}(s) \cdot P_{ii}(s)$

$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)} \quad \forall\, 0 \leq s < 1$$

iff $f_{ii} = \sum_{n \geq 0} f_{ii}(n) = \lim_{s \uparrow 1} F_{ii}(s) = 1$

So by Abel's theorem, we have: $\sum_{n \geq 0} p_{ii}(n) = \lim_{s \uparrow 1} P_{ii}(s) = +\infty$ #

Remark:

$$\sum_{n\geq 0} p_{ii}(n) = \text{"expected number of visits in state } i \mid X_0 = i\text{"}$$

Examples:

- simple random walk (in one dimension) with param. p, q

$$p_{00}(2n) \simeq \frac{(4pq)^n}{\sqrt{\pi n}} \qquad \sum_{n\geq 0} p_{00}(2n) = +\infty \; ?$$

iff $p = q = \frac{1}{2}$

$$\begin{cases} \text{if } p=q=\frac{1}{2} : \; p_{00}(2n) \simeq \frac{1}{\sqrt{\pi n}} \text{ so } \sum_{n\geq 0} p_{00}(2n) \text{ diverges} \\ \text{if not } : \; 4pq < 1 \text{ so } \sum_{n\geq 0} p_{00}(2n) \text{ converges} \end{cases}$$

- simple symmetric random walk in two dimensions: $p_{00}(2n) \simeq \frac{1}{\pi n}$

so $\sum_{n\geq 0} p_{00}(2n)$ diverges $\Rightarrow$ recurrent

Positive and null-recurrence

Let $T_i = \inf\{n \geq 1 : X_n = i\}$ first recurrence time to state i

$f_{ii} = \mathbb{P}(T_i < +\infty \mid X_0 = i)$ [$= 1$ iff state i is recurrent]

$f_{ii} = \sum_{n \geq 1} f_{ii}(n) = \sum_{n \geq 1} \mathbb{P}(T_i = n \mid X_0 = i)$

<u>Def</u>: $\mu_i = \mathbb{E}(T_i \mid X_0 = i)$ mean recurrence time

- if i is transient, then $\mathbb{P}(T_i = +\infty \mid X_0 = i) > 0$, so $\mu_i = +\infty$
- if i is recurrent, then $\mu_i = \mathbb{E}(T_i \mid X_0 = i) = \sum_{n \geq 1} n \cdot \mathbb{P}(T_i = n \mid X_0 = i)$

$$= \sum_{n \geq 1} n \cdot f_{ii}(n) \quad \in [1, +\infty]$$

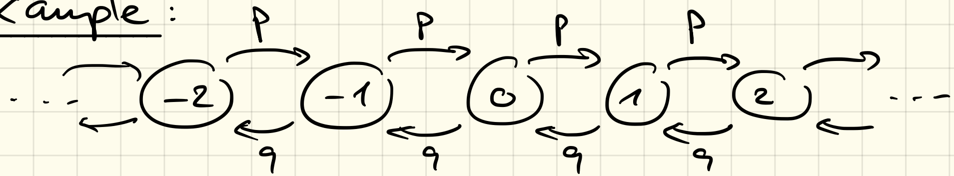
<u>Def</u>: • i is positive-recurrent if $\mu_i < +\infty$

• i is null-recurrent if $\mu_i = +\infty$

Facts:

- In a given equivalence class, either all states are transient, or all states are positive-recurrent, or all states are null-recurrent.
- A finite irreducible is always positive-recurrent.

Example:



$p+q=1$

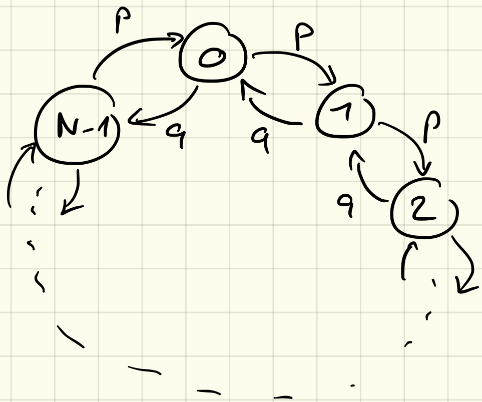
if $p \neq q$: transient chain $\longrightarrow$ $\mu_0 = \mathbb{E}(T_0 \mid X_0 = 0) = +\infty$

$(\mathbb{P}(T_0 = +\infty \mid X_0 = 0) > 0)$

if $p = q = \frac{1}{2}$: recurrent chain $\longrightarrow$ $\mathbb{P}(T_0 = +\infty \mid X_0 = 0) = 0$

but $\mu_0 = +\infty$ null-recurrent

Example: cyclic random walk



$p + q = 1$

$0 < p, q < 1$

$S = \{0, \ldots, N-1\}$ finite

chain irreducible

$\Rightarrow$ all states are positive-recurrent ($\mu_0 < +\infty$)

Q: $\mu_0 = ?$