MCAA lecture 3

Stationary distribution

Distribution at time n : $\pi_j^{(n)} = \mathbb{P}(X_n = j) \quad n \in \mathbb{N},\ j \in S$

$$\pi_j^{(n+1)} = \sum_{i \in S} \pi_i^{(n)} \cdot p_{ij} \quad \forall j \in S$$

in vector form : $\underbrace{\pi^{(n+1)}}_{\text{raw vector}} = \underbrace{\pi^{(n)}}_{\text{raw vector}} \cdot \underbrace{P}_{\text{matrix}}$

Definition: A (probability) distribution $\pi = (\pi_i,\ i \in S)$ $\left[0 \leq \pi_i \leq 1,\ \sum_{i \in S} \pi_i = 1\right]$ is a stationary distribution for the Markov chain X if $\pi_j = \sum_{i \in S} \pi_i \cdot p_{ij} \quad \forall j \in S$

i.e. $\boxed{\pi = \pi \cdot P}$

Implications:

- if π is stationary, then $\pi \cdot P^n = \underbrace{\pi \cdot P}_{\pi} \cdot P^{n-1} = \pi P^{n-1} = \dots \pi$
- if $\pi^{(0)} = \pi$ (= stat. dist.), then $\pi^{(n)} = \pi^{(0)} \cdot P^n = \pi \cdot P^n = \pi \quad \forall n \in \mathbb{N}$

Remarks:

- A stationary distribution is a solution of a system of linear equations; it is not necessarily the case that $\lim_{n\to\infty} \pi^{(n)} = \pi$
- π may not exist in some cases
- π may not be unique in some other cases
- practical remark: in the system of N equations $\pi = \pi \cdot P$ (assume $|S| = N$), there is always one redundant equation; in order to determine π, we need to use also the condition $\sum_{i \in S} \pi_i = 1$.

Mathematical remark:

- Define $\mathbb{1}$ = "all-ones" column vector

  Then $P \cdot \mathbb{1} = 1 \cdot \mathbb{1}$ (i.e. $\sum_{j \in S} p_{ij} = 1 \quad \forall i \in S$)

  "stochastic matrix"

  So $\mathbb{1}$ is an eigenvector of the matrix P (on the right) with corresponding eigenvalue 1.

- Now if there exists a row vector π s.t. $\pi = \pi \cdot P$, then π is also an eigenvector of P (on the left) with the same eigenvalue 1.

Theorem [without proof]

Let X be an irreducible Markov chain.

Then X is positive-recurrent iff

X admits a stationary distribution π

In addition, in this case, if π exists, then it is unique and given by $\pi_i = \frac{1}{\mu_i} = \frac{1}{\mathbb{E}(T_i \mid X_0 = i)} \quad \forall i \in S$

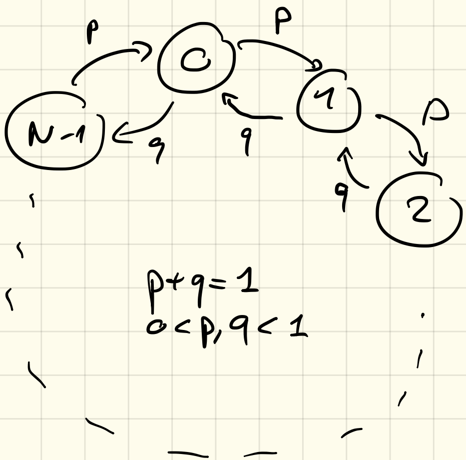
Note: X is positive-recurrent $\Rightarrow \mu_i < +\infty$, so $\pi_i > 0 \quad \forall i \in S$

Corollary: A finite irreducible chain always admits a unique stationary distribution.

Example

Cyclic random walk

$S = \{0, \dots, N-1\}$



$p + q = 1$

$0 < p, q < 1$

finite, irreducible

$\Longrightarrow$ positive-recurrent

$\overset{\text{Thm}}{\Longrightarrow}$ π exists & is unique

$$P = \begin{pmatrix} 0 & p & 0 & \cdots & q \\ q & 0 & p & 0 \cdots & 0 \\ 0 & q & 0 & p & 0 \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ p & \cdots & & q & 0 \end{pmatrix}$$

- $\sum_{j \in S} p_{ij} = 1 \quad \forall i \in S$
- $\sum_{i \in S} p_{ij} = 1 \quad \forall j \in S$

"doubly stochastic matrix"

Proposition

If X is a finite irreducible chain whose transition matrix P is doubly stochastic, then it admits a unique stationary distribution π and π is uniform: $\pi_i = \frac{1}{N} \quad \forall i \in S \quad (|S| = N)$

Proof: Plug $\pi_i = \frac{1}{N}$ into the equation $\pi = \pi P$:

$$\frac{1}{N} = \sum_{\substack{\cancel{j} \in S \\ i}} \frac{1}{N} P_{ij} \quad \forall i \in S \text{ ?}$$

$$1 = \sum_{\substack{\cancel{j} \in S \\ i}} P_{ij} \quad \forall i \in S \text{ ?} \quad \checkmark \quad \text{because } P \text{ is doubly stochastic} \quad \#$$

(uniqueness guaranteed by the thm)

Back to the example

- So $\pi_i = \frac{1}{N} \quad \forall i \in \{0, \dots, N-1\}$

  The thm also says that $\pi_i = \frac{1}{\mu_i}$ so $\underline{\mu_i = N \ \forall i}$

- So $\pi =$ uniform is the "stationary" distribution of the chain

when $p \neq q$, a rotation occurs permanently in one direction or the other $\Rightarrow$ not "truly" stationary

Counter-example

Symmetric simple random walk on $\mathbb{Z}$:

irreducible, recurrent but null-recurrent

Let us prove that the chain is null-reccurrent using the theorem: look for a stationary distribution π:

$$\pi = \pi P \quad \text{i.e.} \quad \forall i \in \mathbb{Z} \quad \pi_i = \frac{1}{2}(\pi_{i-1} + \pi_{i+1})$$

$\Longrightarrow \pi_i = \pi_j \quad \forall i,j \in \mathbb{Z} \longrightarrow$ problem!

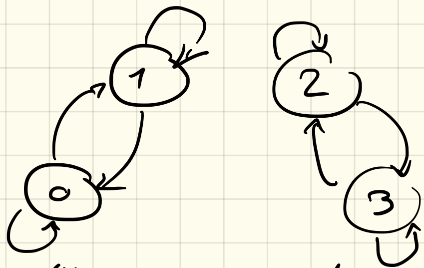
The uniform distribution does not exist on $\mathbb{Z}$!

$\Longrightarrow \pi$ does not exist $\underset{\text{Thm}}{\Longrightarrow} X$ is not positive-reccurrent

$\Longrightarrow X$ is null-recurrent. #

What if the chain is not irreducible?

- two positive-recurrent classes:



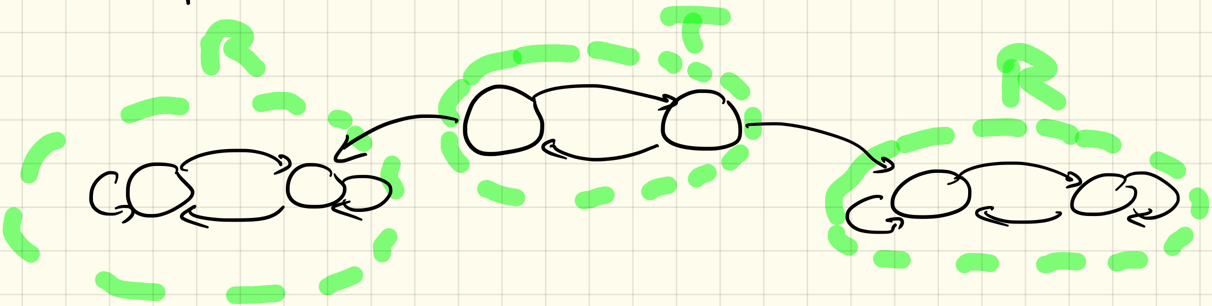
$\pi^{(A)} = \left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad$ $\pi^{(B)} = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow$ a stationary distribution exists but is not unique!

$$\Rightarrow \begin{cases} \pi = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \quad \pi = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), \quad \pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\ \pi = \left(\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) \quad 0 \le \alpha \le 1 \end{cases}$$

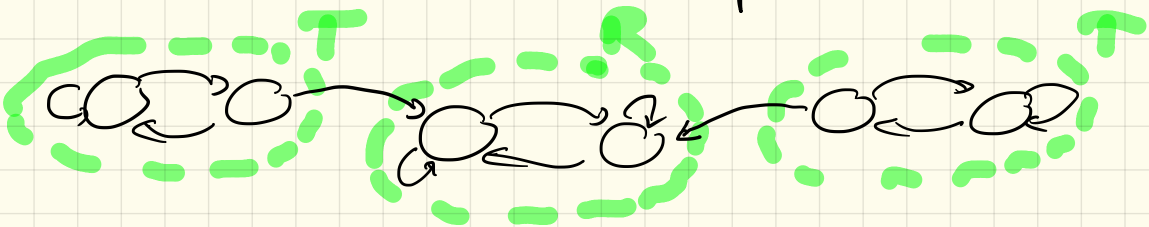
are stationary distributions of the chain

- two positive-recurrent classes and one transient class:



π exists but is not unique: $\pi = \left(\frac{\alpha}{2}, \frac{\alpha}{2}, 0, 0, \frac{1-\alpha}{2}, \frac{1-\alpha}{2} \right)$

- two transient classes and one positive-recurrent class:



π exists and is unique: $\pi = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right)$

Limiting distribution

Definition: A distribution π is a limiting distribution for the Markov chain $(X_n, n \geq 0)$ if $\forall$ initial distribution $\pi^{(0)}$, $\lim_{n\to\infty} \pi^{(n)} = \pi$

Remarks:

- such a limiting distribution may not exist
- but if it exists, then it is unique!
- if π is a limiting distribution, then it is a stationary dist.

Proof:

$$\begin{array}{ccc} \pi^{(n+1)} & = & \pi^{(n)} \cdot P \quad \forall n \in \mathbb{N} \\ {\scriptstyle n\to\infty}\downarrow & & \downarrow{\scriptstyle n\to\infty} \\ \pi & = & \pi \cdot P \end{array}$$

"#" (⚠ $|S| = +\infty$ case)

Def: A Markov chain is ergodic if it is irreducible, aperiodic and positive-recurrent.

Ergodic theorem

Let X be an ergodic Markov chain.

Then it admits a unique limiting and stationary distribution π, i.e.:

- $\forall \pi^{(0)}, \lim_{n\to\infty} \pi^{(n)} = \pi$ $\iff$ $\forall i,j \in S$ $\lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j$
- $\pi = \pi P$

__Remark__ : __aperiodicity matters__ !

__Ex__ : consider the chain

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \end{tikzcd} \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

periodicity = 2

stationary distribution? $\pi = \pi \cdot P \longrightarrow \pi = \left(\frac{1}{2}, \frac{1}{2}\right)$

is __the__ solution

limiting distribution?

if $\pi^{(0)} = (1, 0)$, then $\pi^{(1)} = (0, 1)$, $\pi^{(2)} = (1, 0)$ ...

so $\lim_{n \to \infty} \pi^{(n)}$ does not exist!

Modified ex:

$$\begin{tikzcd} 0 \arrow[loop left, "\varepsilon"] \arrow[r, bend left, "1-\varepsilon"] & 1 \arrow[l, bend left, "1"] \end{tikzcd} \qquad \varepsilon > 0 \qquad P = \begin{pmatrix} \varepsilon & 1-\varepsilon \\ 1 & 0 \end{pmatrix}$$

$\left.\begin{array}{l} \underbrace{\text{finite, irreducible}}_{\Rightarrow \text{ positive-recurrent}}\text{, aperiodic chain} \end{array}\right\} \Rightarrow$ ergodic

$\Rightarrow \exists! \, \pi =$ limiting & stationary distribution ✓

Last remark: So can't we say anything for a periodic chain?
(irreducible & positive-reccurent)

Yes we can!

$$\forall \pi^{(0)}, \quad \frac{1}{n} \sum_{k=1}^{n} \pi_i^{(k)} \xrightarrow[n \to \infty]{} \pi_i \quad \forall i \in S$$