MCAA lecture 4: quiz solutions

1) b) $\nu = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the closest to $\mu = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$:

$$\|\nu - \mu\|_{TV} = \frac{1}{2}\left(\left|\frac{1}{3} - \frac{1}{2}\right| + \frac{1}{3} + \left|\frac{1}{3} - \frac{1}{2}\right|\right) = \frac{1}{3}$$

d) $\nu = (0, 1, 0)$ is the farthest to $\mu = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$: $\|\mu - \nu\|_{TV} = 1$

[and a), c) are both at distance $\frac{1}{2}$]

2) c) $A = \{1, 2\}$ is the answer: $\|\mu - \nu\|_{TV} = \frac{1}{2}\left(\frac{1}{2} + 0 + \frac{1}{2}\right) = \frac{1}{2}$

and $\mu(A) - \nu(A) = \frac{1}{2} - 0 = \frac{1}{2}$

(Note that $A = \{2\}$ would also work, and that $A = \{0, 1\}$ gives $\mu(A) - \nu(A) = \frac{1}{2} - 1 = -\frac{1}{2}$, so $|\mu(A) - \nu(A)| = \|\mu - \nu\|_{TV}$)

3) a), d), e) and f) are couplings of μ & ν:

a) is the "grand coupling": $\mathbb{P}(X=Y)=1$

e) is the "statistical coupling": X & Y are independent

d) is something inbetween (positive correlation between X & Y)

f) is the case where X, Y are the most negatively correlated as possible

For b) and c), computing the marginals $\mu_0 = p_{00} + p_{01}$ etc. does not lead to the desired values for μ & ν.

<u>Subsidiary question</u>: $\|\mu - \nu\|_{TV} = 0$, and the only coupling for which $\mathbb{P}(X \neq Y) = 0$ is the "grand coupling" a).