Homework 7

Exercise 1. Let $d \ge 1$ and m > 2 be integers, and let us consider the following process on $S = \{0, \ldots, m-1\}^d$: at each step, from state $x \in S$, pick a component of x uniformly at random and change it to *another* number in $\{0, \ldots, m-1\}$, chosen again uniformly at random.

a) Write down the transition matrix P of this chain. Is this chain is ergodic? What is its stationary distribution? Is the detailed balance equation satisfied?

It turns out that the eigenvectors of P are given by $(\phi^{(z)}, z \in S)$, where

$$\phi_x^{(z)} = \exp(2\pi i x \cdot z/m), \quad x \in S,$$

and $x \cdot z = \sum_{j=1}^d x_j z_j$.

b) Compute the corresponding eigenvalues $(\lambda_z, z \in S)$ of P.

Hint: Express these in terms of $|z| = \sharp \{ \text{ non-zero components of } z \}$.

c) Deduce the value of the spectral gap for d > 2, as well as a corresponding upper bound on $||P_0^n - \pi||_{\text{TV}}$.

d) Compare this upper bound to the general lower bound found in class. Do these two bounds match for large m and d?

e*) Through a more careful analysis, find a tighter upper bound on $||P_0^n - \pi||_{\text{TV}}$ for large m and d.

Exercise 2. Regarding the lazy random walk on $\{0,1\}^d$, we saw in class that $||P_0^n - \pi||_{\text{TV}}$ is arbitrarily close to 1 for

$$n = \frac{d+1}{4} \left(\log d - c \right)$$

and c > 0 arbitrarily large. Following the reasoning made in class (but the technique is simpler here!), show that the following distance:

$$||P_0^n - \pi||_2 = \left(\sum_{y \in \{0,1\}^d} \left(\frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y}\right)^2\right)^{1/2}$$

can be made arbitrarily large by taking again $n = \frac{d+1}{4} (\log d - c)$ and c > 0 arbitrarily large.

NB: The above distance is the ℓ^2 -distance between P_0^n and π ; it has been shown in class to be an upper bound on $\|P_0^n - \pi\|_{\text{TV}}$ (with an extra factor 2).