LECTURE 1

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1. Review of topological notions

Let X be a set. A *topology* on X is a collection of subsets $\mathcal{T}$ of X, called *open sets* satisfying:

(1) X and $\emptyset$ are in $\mathcal{T}$.
(2) $\mathcal{T}$ is closed under the operation of unions: unions of open sets are open.
(3) $\mathcal{T}$ is closed under the operation of finite intersections: any intersection of finitely many open sets is open.

A space with the above structure is called a topological space. A collection of subsets $\mathcal{B}$ of X is called a *basis* for $\mathcal{T}$, if:

(1) $\mathcal{B} \subseteq \mathcal{T}$.
(2) Every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$

More generally, given a set X (without a topology on it, a priori), a collection of subsets $\mathcal{B}$ of X is called an abstract *basis* if:

(1) $X = \bigcup_{B \in \mathcal{B}} B$ and $\mathcal{B}$ contains the emptyset.
(2) For each triple x, B_1, B_2, where $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Given an abstract basis, the collection of unions of sets in the basis is the topology generated by the basis, and it satisfies the axioms above (prove it!).

An open set that contains a point $p \in X$ is called a *neighbourhood* of p. A set U is said to be *closed*, if the complement $U^c = X \setminus U$ is open. The family of closed sets of a topological space satisfies the following:

(1) X and $\emptyset$ are closed.
(2) Intersections of closed sets are closed.
(3) Any union of finitely many closed sets is closed.

Given a topology $\mathcal{T}$ on a set X, and a point $p \in X$, a *neighbourhood basis at* p is a collection $\mathcal{B}_p$ of neighbourhoods of p such that every neighbourhood of p contains as a subset an element of $\mathcal{B}_p$. $(X, \mathcal{T})$ is said to be *first countable*, if there is a countable neighbourhood basis at each point. $(X, \mathcal{T})$ is said to be *second countable*, if it admits a countable basis for the topology $\mathcal{T}$.

The interior of a set $S \subseteq X$, denoted by $Int(S)$, is the union of all open subsets of X contained in S, and hence itself is open. The exterior of S, $Ext(S)$, is the interior of the complement of S. The closure of S, denoted as $\bar{S}$, is the intersection of all closed sets containing S, hence itself is closed. S is *dense* in X if $\bar{S} = X$. S is *nowhere dense* in X if $Int(\bar{S}) = \emptyset$. The boundary of S, denoted as δS, is $\bar{S} \setminus Int(S)$. A point $p \in S \subseteq X$ is an *isolated point in* S, if there is an open set U such that $U \cap S = \{p\}$. A point $p \in X$ is a limit point of a set $S \subset X$, if for each open neighborhood U of p, $U \cap S \setminus \{p\} \neq \emptyset$. A sequence $\{p_n\}_{n \in \mathbf{N}}$ of points in X is said to converge to a point $p \in X$, if for each neighbourhood U of p, there is a number $N \in \mathbf{N}$ such that $\forall n > N$, $p_n \in U$.

Let X, Y be topological spaces. A map $f : X \to Y$ is said to be *continuous*, if for every open set $U \subset Y$, $f^{-1}(U)$ is open. The map f is called a *homeomorphism* if it is a continuous bijection with a continuous inverse. The map f is called a *local homeomorphism* if the restriction of f to some open set U of X is a homeomorphism onto its image.

A metric space (X, d) is a set X with a distance function $d : X \times X \to \mathbf{R}_{\geq 0}$ satisfying the following for all $x, y, z \in X$:

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

Given a metric space, we define the open and closed balls of radius $r > 0$, around a point $p \in X$ as follows:

$$B_r(x) = \{y \in X \mid d(y, x) < r\} \qquad \bar{B}_r(x) = \{y \in X \mid d(y, x) \leq r\}$$

The set of open balls of a metric space satisfy the axioms of an abstract basis for a topology. A topological space is said to be *metrizable*, if it admits a metric whose open balls generate a topology that coincides with the given topology. The *diameter* of a set $S \subseteq X$ is the quantity $Sup\{d(x,y) \mid x, y \in S\}$. The set S *bounded* if the diameter is finite.

A topological space X is said to be *Hausdorff* if the following holds. For each pair of points $x, y \in X$, there exist neighbourhoods U of x and V of y, that are disjoint. Metric spaces are easily seen to be Hausdorff, for instance.

Exercise 1.1. *Show that in a Hausdorff space, the following holds:*

(1) *Limits of convergent sequences are unique.*
(2) *Finite subsets are closed.*

Given a topological space, an *open covering* is a collection of open sets whose union is the whole space. A subcovering is a subcollection of an open covering that is itself a covering. A topological space is said to be *compact*, if every open covering of the space admits a finite subcovering. A topological space is said to be *second countable*, if the topology admits a countable basis.

Exercise 1.2. *Show that second countable spaces must satisfy that every open covering has a countable subcovering.*

Given a subset $S \subset X$ of a topological space X, the *induced topology* on S is defined as follows. A subset $V \subset S$ is open if and only if there is an open set $U \subset X$ such that $U \cap S = V$. Similarly, a subset $V \subset S$ is closed if and only if there is a closed set $U \subset X$ such that $U \cap S = V$. An injective map $f : X \to Y$ between topological spaces is said to be a *topological embedding*, if it is a homeomorphism onto its image.

Product topology Let $\{X_\alpha \mid \alpha \in I\}$ be a finite collection of topological spaces. The product space $X = \prod_{\alpha \in I} X_\alpha$ is endowed with the topology with the following basis of open sets. The open sets in this basis are all sets of the following form:

$$V = \prod_{\alpha} V_\alpha \qquad V_\alpha \text{ is open in } X_\alpha$$

Theorem 1.3. *(Tychonoff's theorem) A product of compact topological space is also compact.*

Quotient topology Let X be a topological space, Y a set, and $\phi : X \to Y$ a surjective map. The *quotient topology* on Y is defined as follows. A subset $U \subseteq Y$ is open if and only if $\phi^{-1}(U)$ is open in X.

Proposition 1.4. *(Characteristic property of the quotient topology) Let X, Z be topological spaces, Y a set, and $\phi : X \to Y$ a surjective map. Consider the quotient topology on Y. A map $\tau : Y \to Z$ is continuous if and only if $\tau \circ \phi$ is continuous.*