LECTURE 1

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1. Topological manifolds

A topological space M is called a *topological manifold of dimension n* if it satisfies the following:

(1) It is Hausdorff.
(2) It is second countable.
(3) (Locally Euclidean) For each point $x \in M$, there is a neighbourhood U of x which is homeomorphic to an open subset of $\mathbf{R}^n$. Hence, there is an open set U containing x, an open set $V \subset \mathbf{R}^n$, and a homeomorphism $\phi : U \to V$.

The following is a theorem that we shall not prove in this course, but it is important to know.

Theorem 1.1. *(Topological invariance of dimension) If two manifolds are homeomorphic, then they must have the same dimension.*

Given an n-dimensional topological manifold M, a *coordinate chart*, or simply a *chart*, consists of the following:

(1) An open set $U \subset M$, called the *coordinate domain*.
(2) An open set $V \subset \mathbf{R}^n$.
(3) A homeomorphism $\phi : U \to V$, called the *coordinate map*, with *component functions* defined as follows:

$$(x^1, ..., x^n) \qquad x^i : U \to \mathbf{R} \qquad \phi(p) = (x^1(p), ..., x^n(p))$$

Note that by definition, each point $x \in M$ is contained in the domain of some coordinate chart. The following convention shall be useful. If $p \in U$ and $\phi(p) = 0 \in \mathbf{R}^n$, then we say that the chart is *centered at p*. It is also common to denote a chart by its coordinate functions as either of the following:

$$(U, (x^1, ..., x^n)) \qquad (U, (x^i))$$

1.1. Examples of topological manifolds.

Now we shall discuss five important examples.

(1): Graphs of continuous functions

Let $U \subset \mathbf{R}^n$ be an open set, and let $\phi : U \to \mathbf{R}^m$ be a continuous map. Consider the set

$$\Gamma(f) = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid x \in U, y = \phi(x)\}$$

endowed with the subspace topology. Let $\pi_1 : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be the projection to the first factor. We define

$$\phi : \Gamma(f) \to \mathbf{R}^n = \pi_1 \restriction \Gamma(f)$$

Since ϕ is a continuous bijection with a continuous inverse, it is a homeomorphism. It follows that $\Gamma(f)$ is a manifold of dimension n with a single *global* coordinate chart ϕ.

(2): Sphere

Exercise 1.2. *Show that subspaces of second countable spaces are second countable. Show the same for the Hausdorff property.*

For each $n \in \mathbf{N}$, the sphere $\mathbf{S}^n$ is Hausdorff and second countable, since it is a subspace of $\mathbf{R}^{n+1}$. To see that it is a manifold, we need to show that it is locally Euclidean. For each $1 \leq i \leq n+1$, we define the subspaces

$$X_i^+ = \{(x_1, ..., x_{n+1}) \in \mathbf{S}^{n+1} \mid x_i > 0\} \qquad X_i^- = \{(x_1, ..., x_{n+1}) \in \mathbf{S}^{n+1} \mid x_i < 0\}$$

For each $1 \leq i \leq n+1$, we check that $X_i^\pm$ are graphs of the functions

$$f_i^\pm : \mathbf{B}^n \to \mathbf{R} \qquad f_i^{\pm 1}(v) = \pm\sqrt{1 - |v|^2} \qquad v = (x_1, ..., \hat{x_i}, ..., x_{n+1})$$

where the $\hat{x_i}$ denotes that this coordinate has been removed. Here $\mathbf{B}^n$ is the open n-dimensional unit ball in $\mathbf{R}^n$ centered at 0. These $X_1^\pm, ..., X_{n+1}^\pm$ provide the required charts, making our topological space locally Euclidean.

(3): Projective spaces

The n-dimensional real projective space is denoted as $\mathbf{P}^n$. This is the space of all 1-dimensional linear subspaces of $\mathbf{R}^{n+1}$, with the quotient topology induced by the natural map

$$\tau : \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{P}^n \qquad \tau(x) = [x]$$

where $[x]$ denotes the linear subspace of $\mathbf{R}^{n+1}$ spanned by x.

For each $1 \leq i \leq n+1$, let

$$U_i = \tau(V_i) \qquad V_i = \{(x_1, ..., x_{n+1}) \in \mathbf{R}^{n+1} \mid x_i \neq 0\}$$

Define the map

$$\phi_i : V_i \to \mathbf{R}^n \qquad \phi_i(x_1, ..., x_{n+1}) = (\frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i})$$

The map ϕ_i is well defined, since it is invariant under multiplying the input vector $(x_1, ..., x_{n+1})$ by a scalar. Moreover, ϕ_i is a bijection with inverse $(x_1, ..., x_n) \to (x_1, ..., x_{i-1}, 1, x_i, ..., x_n)$. The map ϕ_i is seen to be a homeomorphism thanks to the characteristic property of quotient maps. The domains of these maps cover the projective plane, and hence provide the coordinate charts.

Exercise 1.3. *Show that the projective plane is Hausdorff and second countable. Show that it is compact.*

(4): Product manifolds and the Torus

Exercise 1.4. *Show that a finite product of Hausdorff and second countable spaces is also Hausdorff and second countable.*

Let $M_1, ..., M_k$ be topological manifolds of dimension $n_1, ..., n_k$ respectively. The product space $M = M_1 \times ... \times M_k$ is a manifold of dimension $n_1 + ... + n_k$. Thanks to the exercise above, it is second countable and Hausdorff. We need to check that it is locally Euclidean. Given a point $(p_1, ..., p_k) \in M$, we know that there exist charts for each $1 \leq i \leq k$

$$\phi_i : U_i \to \bar{U}_i \subset \mathbf{R}^{n_i} \qquad p_i \in U_i \subset M_i$$

In effect, we obtain a chart

$$\phi_1 \times ...\phi_k : U_1 \times ...U_k \to \bar{U}_1 \times ... \times \bar{U}_k \qquad \bar{U}_1 \times ... \times \bar{U}_k \subset \mathbf{R}^{n_1 + .. + n_k}$$

An example of a product manifold is the n-torus, which is the n-manifold:

$$\mathbf{T}^n = \mathbf{S}^1 \times ... \times \mathbf{S}^1 \text{ (an } n\text{-fold product)}$$

Exercise 1.5. *Consider the topological space constructed as follows. Let*

$$X_1 = \mathbf{R} \times \{a\} \qquad X_2 = \mathbf{R} \times \{b\}$$

where a, b is a set of two points. Note that X_1, X_2 are both homeomorphic to $\mathbf{R}$. Consider the disjoint union of X_1, X_2. Now X is the quotient space given by identifying pairs $(x, a) \sim (x, b)$ whenever $x \neq 0$. Show that X is locally Euclidean and second countable, but not Hausdorff. Hence, it is not a manifold.

2. Properties of topological manifolds

Recall that an open set U of a topological space X is *precompact* if its closure is compact.

The following Lemma is fundamental and important.

Lemma 2.1. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. **Step** 1: Show this for each coordinate chart. Let (ϕ, U) be a coordinate chart. For each pair $x \in \phi(U), r \in \mathbf{Q}$ such that $B_r(x) \subset \phi(U)$, we consider the coordinate ball $\phi^{-1}(B_r(U))$. These are clearly precompact in U, since ϕ is a homeomorphism. Their closure in U is the same as their closure in M, since M is Hausdorff (prove it!). So they are precompact in M. These coordinate balls form a basis for U, since ϕ is a homeomorphism.

Step 2: Using second countability, we can cover the manifold with a countable basis of charts. We consider for each chart the coordinate balls coming from Step 1. The union of all such coordinate balls over all such charts provides the required basis. □

Corollary 2.2. *Manifolds are locally compact: every point admits a neighbourhood that is contained in a compact subset.*

Proof. This follows from Lemma 2.1, since coordinate balls are precompact. □

A topological space is *connected* if there do not exist disjoint open sets whose cover is X. It is *path connected* if any pair of points can be joined by a path. It is *locally path connected*, if it has a basis of path connected open sets.

Proposition 2.3. *Let M be a topological manifold. Then the following hold:*

1. *M is locally path connected.*
2. *M is path connected if and only if it is connected.*
3. *The components are the same as the path components.*
4. *M has countably many components, each of which is open in M and a connected topological manifold.*

Proof. Use Lemma 2.1. Proof in class. □

A topological space is said to be *paracompact*, if every open covering admits a locally finite subcovering: this means that for each point, only finitely many open sets in the subcovering contain the point.