LECTURE 2

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1. Smooth manifolds

A topological manifold manifold is called a *smooth manifold*, if it is endowed with the following additional structure.

Definition 1.1. (Smooth manifold) Let M be a topological n-manifold. Two charts $(\phi_1, U_1), (\phi_2, U_2)$ are said to be *smoothly compatible* if the maps

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \qquad \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

are smooth maps (i.e. C^∞). A *smooth atlas* is a family of charts that are smoothly compatible, and that cover M. A smooth atlas is called *maximal*, if it is not contained in a strictly larger smooth atlas. A maximal smooth atlas is called a *smooth structure* on M. A *smooth manifold* is a topological manifold endowed with a smooth structure. The charts in the smooth structure are called *smooth charts*.

Proposition 1.2. *Every smooth atlas can be extended uniquely to a maximal smooth atlas.*

Proof. Let M be a topological manifold, and let $\mathcal{A}_1$ be a smooth atlas. Let $\mathcal{A}$ be a family of charts that contains each chart in $\mathcal{A}_1$, and moreover contains each chart which is smoothly compatible with each chart in $\mathcal{A}_1$. We claim that $\mathcal{A}$ is a smooth atlas. The fact that it is unique and maximal follows immediately from the definition of $\mathcal{A}$.

Let $(\phi_1, U_1), (\phi_2, U_2)$ be charts in $\mathcal{A}$. We need to show that for each $x \in \phi_2(U_1 \cap U_2)$, the map

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$$

is smooth at x. Let (ν, V) be a chart in $\mathcal{A}_1$ such that $\phi_2^{-1}(x) \in (V)$.

Since $\phi_1 \circ \nu^{-1}, \nu \circ \phi_2^{-1}$ are smooth on their domains of definition, so is their composition which equals $\phi_1 \circ \phi_2^{-1}$. This proves the Proposition. □

The smooth structure $\mathcal{A}$ above is said to be *generated by the smooth atlas* $\mathcal{A}_1$. In practise, maximality of the structure is usually not that important.

Example 1.3.
1. Any manifold with only one chart, for instance $\mathbf{R}^n$ or a graph of a continuous function, is a smooth manifold.
2. The sphere and the projective plane from the previous lecture are also smooth manifolds.

Definition 1.4. (Smooth maps) Let M, N be smooth manifolds. A map $F : M \to N$ is a *smooth map* if for each point $x \in M$, there is a pair of smooth charts (ϕ_1, U_1) for M and (ϕ_2, U_2) for N, such that

$$x \in U_1 \qquad F(U_1) \subset U_2$$

and the map

$$\hat{F} = \phi_2 \circ F \circ \phi_1^{-1} : \phi_1(U_1) \to \phi_2(U_2)$$

is a smooth map. The map $\hat{F}$ is called a *coordinate representation* of F.

We remark that the condition $F(U_1) \subset U_2$ is included in the above definition, because this automatically implies continuity of the map F.

Recall that a topological manifold admits a basis of coordinate balls. In the case of a smooth manifold, one can similarly prove the existence of a basis of *smooth* coordinate balls. Let M be a n-dimensional smooth manifold. We say that a subset $U_1 \subset M$ and a smooth chart $\phi_1 : U \to \mathbf{R}^n$ is a *regular coordinate ball*, if the following holds. There is a set $U_1 \subset U_2 \subset M$ and a smooth chart $\phi_1 : U_2 \to \mathbf{R}^n$, and numbers $r_1 < r_2$ such that

$$0 \in \phi_1(U_1) \subset \overline{\phi_1(U_1)} \subset \phi_2(U_2)$$

Indeed, we have the following:

Proposition 1.5. *Every smooth manifold admits a countable basis of regular coordinate balls.*

Note that every regular coordinate ball is precompact in M.

Example 1.6. (A non-example) Consider the standard chart on $\mathbf{R}$ (the identity map), and the chart given by $\phi(x) = x^3$. The transition function $id \circ \phi^{-1} = x^{\frac{1}{3}}$ is not smooth at 0.

Example 1.7. We denote by $M(n \times m, \mathbf{R})$ as the set of $n \times m$ matrices with real entries. This is in natural bijective correspondence with $\mathbf{R}^{n \times m}$, and hence naturally a smooth manifold. The same for $M(n \times m, \mathbf{C})$. We usually denote $M(n \times n, \mathbf{R})$ as $M(n, \mathbf{R})$.

Example 1.8. (Open submanifolds) Let U be an open subset of a smooth manifold M. We define the following atlas on U:

$$\mathcal{A} = \{(V, \phi) \text{ smooth charts in the smooth atlas for } M \mid V \subset U\}$$

Note that since the intersection of a domain of a smooth chart with U, is a smooth chart for U, it is easy to show that the above is a smooth atlas.

A specific example is the set of invertible matrices $GL(n, \mathbf{R})$, which form an open subset of $M(n, \mathbf{R})$.

Definition 1.9. (Einstein summation convention) Given a sum of monomial terms, we may remove the summation sign, and interpret the sum as follows. Whenever the same index appears twice in a monomial term, once as an upper index and once as a lower index, we interpret that the sum is over that index. For example, we may write $\sum_{1 \leq i \leq n} x^i E_i$ as simply $x^i E_i$.

Another aspect of this notion for vector spaces is as follows. We usually write the components as x^i, i.e. with upper indices, and basis vectors as E_i, with lower indices.

Example 1.10. (Linear maps) Let V be an n-dimensional vector space. We endow V with the topology induced by the inner product. (Actually, the topology is independent of the choice of inner product). Each ordered basis $E_1, ..., E_n$ for V describes an isomorphism $E : \mathbf{R}^n \to V$ as

$$E(x) = x^i E_i \qquad x = (x^1, ..., x^n)$$

Note that the above equality follows the *Einstein summation convention.* For now, we simply drop the summation sign when the context is clear. Note that (V, E^{-1}) is a chart for V, since E is also a homeomorphism.

Let $B_1, ..., B_n$ be another ordered basis, and

$$B : \mathbf{R}^n \to V \qquad B(x) = x^i B_i$$

be the corresponding isomorphism. Then there is an invertible matrix $A = (A^i_j)$ such that the transition map

$$B^{-1} \circ E(x^j) = B^{-1}(x^j E_j) = (x^i A^1_i, x^i A^2_i, ..., x^i A^n_i) = \sum_{1 \leq j \leq n} (x^i A^j_i) B_j$$

Since the transition maps are smooth, the set of all such isomorphisms defines a smooth structure on V.

Exercise 1.11. *Let V_1, V_2 be finite dimensional vector spaces. Let $L(V_1, V_2)$ be the set of linear maps from V_1 to V_2. Describe the natural topology, and smooth structure on this space.*

Exercise 1.12. *Show that the charts given in the previous lecture for the sphere and the projective plane at smooth charts. Show the same for products of smooth manifolds, thereby providing a smooth structure on the Torus.*

1.1. **Building manifolds from scratch.** Note that so far to produce a smooth structure, we have to do a fair amount of work. First we have to start with a topological manifold, checking along the way that it satisfies the axioms. Then we endow it with a smooth structure. The following Lemma provides a setup to construct a smooth manifold in one step.

Lemma 1.13. *(Smooth manifold chart Lemma) Let M be a set and let U_α be a collection of subsets of M, V_α be a collection of open subsets of $\mathbf{R}^n$, for $\alpha \in I$, and bijective maps*

$$f_\alpha : U_\alpha \to V_\alpha$$

such that the following are satisfied:

(1) *For each $\alpha, \beta \in I$, the sets $f_\alpha(U_\alpha \cap U_\beta), f_\beta(U_\alpha \cap U_\beta)$ are open.*
(2) *For each $\alpha, \beta \in I$, the transition maps*

$$f_\alpha \circ f_\beta^{-1} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$$

are smooth.

(3) *Countably many U_α cover M.*

(4) *For any pair of points $x, y \in M$, either there is an $\alpha \in I$ such that $x, y \in U_\alpha$, or there are $\alpha, \beta \in I$ such that*
$$U_\alpha \cap U_\beta = \emptyset \qquad x \in U_\alpha, y \in U_\beta$$

We define the basis for the topology as the set
$$\{f_\alpha^{-1}(B) \mid \alpha \in I, B \text{ is an open ball in } f_\alpha(U_\alpha) = V_\alpha\}$$

Then M has a unique smooth manifold structure with this topology such that each pair (f_α, U_α) is a smooth chart.

Proof. It is easy to check that the topology generated by this basis is Hausdorff. We leave it as an exercise (see below) that the topology is second countable. We can easily verify that the charts are Locally Euclidean and smooth. □

Exercise 1.14. *Let X be a topological space, and suppose X admits a countable open cover $\{U_i\}_{i \in I}$ such that each set U_i is second countable in the subspace topology. Show that X is second countable.*

1.2. **The Grassmann manifold.** So far the constructions of manifolds have been rather straightforward. Now we shall construct a smooth manifold using the chart lemma, which will be a bit more involved.

Let V be an n-dimensional vector space. We denote by $G_k(V)$ as the set of k-dimensional linear subspaces of V. When V is $\mathbf{R}^n$, we simply denote $G_k(V)$ as $G_{k,n}$. Let Q be an $(n-k)$-dimensional linear subspace of V. We denote by $\mathcal{U}_Q \subset G_k(V)$ as the set of k-dimensional linear subspaces of V, i.e. elements of $G_k(V)$, who intersection with Q is the zero subspace. These subsets will provide the charts for an application the chart lemma, via the following.

Lemma 1.15. *Let P, Q be linear subspaces of V such that $dim(P) = k, dim(Q) = n-k$ and $V = P \oplus Q$. The k-dimensional linear subspaces of V that have the property that their intersection with Q is the zero subspace, are precisely the graphs of linear maps $P \to Q$.*

Proof. Let $F : P \to Q$ be a linear map. It is clear that the graph of the linear map satisfies the required property. To see the converse, note that if U is a k-dimensional linear subspace of V with the property that its intersection with Q is the zero subspace, the projections of $\pi_P : U \to P$ and $\pi_Q : U \to Q$ induce a well define linear map $F : P \to Q$ defined as follows. The projection $\pi_P : U \to P$ must be an isomorphism, or else the kernel will have dimension greater than 0, and since the kernel is precisely the intersection with Q, this is impossible. So it has a well defined inverse isomorphism, which we denote as $\sigma_{P,U} : P \to U$. Then $F = \pi_Q \circ \sigma_{P,U}$ and the given subspace is the graph. Checking that the correspondence is bijective is left as an easy exercise. □

Recall from the previous exercise that
$$L(P, Q) \cong M((n-k) \times k, \mathbf{R}) \cong R^{k(n-k)}$$
and hence has the natural structure of a smooth manifold. We may regard $(\phi_Q, \mathcal{U}_Q)$ as a coordinate chart for $G_k(V)$.

We denote the maps emerging from the Lemma above as
$$\phi_Q : \mathcal{U}_Q \to L(P, Q) \qquad \Gamma_Q : L(P, Q) \to \mathcal{U}_Q$$
and we fix the isomorphism $\sigma_{P,U} : P \to U$ as the inverse of the projection map $\pi_P : U \to P$, whenever $P \oplus Q = V$ and $Q \cap U$ is the zero subspace. Finally, given $H \in L(P, Q)$, the corresponding graph of H is $\Gamma(H) = H(v) + v$.

Lemma 1.16. *Let $V = P_1 \oplus Q_1 = P_2 \oplus Q_2$ where P_1, P_2 are k-dimensional linear subspaces. Let $(\phi_{Q_i}, \mathcal{U}_{Q_i})$ be the charts, as above. Then*
$$V_1 = \phi_{Q_1}(\mathcal{U}_{Q_1} \cap \mathcal{U}_{Q_2}) \qquad V_2 = \phi_{Q_2}(\mathcal{U}_{Q_1} \cap \mathcal{U}_{Q_2})$$
are open in $L(P_1, Q_1), L(P_2, Q_2)$ respectively. And the transition map
$$\phi_{Q_2} \circ \phi_{Q_1}^{-1} : \phi_{Q_1}(\mathcal{U}_{Q_1} \cap \mathcal{U}_{Q_2}) \to \phi_{Q_2}(\mathcal{U}_{Q_1} \cap \mathcal{U}_{Q_2})$$
is smooth.

Proof. Note that V_1 consists of precisely the elements $F \in L(P_1, Q_1)$ whose graphs have trivial intersection with Q_2. This is true for F if and only if $\pi_{P_2} \circ H$ has trivial kernel, where
$$H(v) = v + F(v), v \in P_1$$
is the corresponding graph of F. In turn, this is true if and only if the matrix of the linear map $\pi_{P_2} \circ H$ is invertible, and has determinant nonzero. Since $\pi_{P_2} \circ H$ depends continuously on F, we obtain that V_1 is open in $L(P_1, Q_1)$. The same follows for V_2 by the symmetric argument.

Now we will show that the transition maps are smooth. Let

$$X_1 \in \phi_{Q_1}(\mathcal{U}_{Q_1} \cap \mathcal{U}_{Q_2}) \qquad X_2 = \phi_{Q_2} \circ \phi_{Q_1}^{-1}(X_1)$$

Note that

$$X_1 : P_1 \to Q_1 \qquad X_2 : P_2 \to Q_2$$

are linear maps. Let

$$U_1 = X_1(v) + v \qquad v \in P_1$$

be the graph of X_1. Then

$$X_2 = \pi_{Q_2} \circ (\pi_{P_2} \restriction U_1)^{-1} = (\pi_{Q_2} \circ \sigma_{P_1,U_1}) \circ (\pi_{P_2} \circ \sigma_{P_1,U_1})^{-1}$$

Finally, we finish by showing that each of the maps

$$(\pi_{Q_2} \circ \sigma_{P_1,U_1}) : P_1 \to Q_2$$

$$(\pi_{P_2} \circ \sigma_{P_1,U_1}) : P_1 \to P_2$$

are compositions of linear maps, i.e.

$$(\pi_{Q_2} \circ \sigma_{P_1,U_1})v = (A + CX_1)v \qquad (\pi_{P_2} \circ \sigma_{P_1,U_1})v = (B + DX_1)v$$

where A, B, C, D are fixed matrices and X_1 is interpreted as an $(n-k) \times k$ matrix. Such maps are clearly smooth with smooth inverses. □

Exercise 1.17. *Provide the details of the last paragraph of the previous proof.*

Exercise 1.18. *Let $e^1, ..., e^n$ be a basis for the vector space V. Show that the sets*

$$\{\mathcal{U}_{Q_J} \mid Q_J \text{ is the subspace spanned by } J \subset \{e^1, ..., e^n\}, |J| = n-k\}$$

cover $G_k(V)$.

Exercise 1.19. *Show that each pair of elements of $G_k(V)$ lie in the chart (ϕ, Q) for some $n-k$ dimensional subspace Q of V.*

2. Manifolds with boundary

We define the upper half space $\mathbf{H}^n$ as

$$\mathbf{H}^n = \{(x^1, ..., x^n) \in \mathbf{R}^n \mid x_n \geq 0\}$$

We also define

$$Int\mathbf{H}^n = \{(x^1, ..., x^n) \in \mathbf{R}^n \mid x_n > 0\}$$

$$\delta\mathbf{H}^n = \{(x^1, ..., x^n) \in \mathbf{R}^n \mid x_n = 0\}$$

Definition 2.1. An n-dimensional *topological manifold with boundary* is a second countable, Hausdorff space, in which every point has a neighbourhood that is either homeomorphic to an open subset of $\mathbf{R}^n$, or to a relatively open subset of $\mathbf{H}^n$. We call the former *interior charts* and the latter *boundary charts.*

A point $x \in M$ is an interior point if it lies in the domain of an interior chart, and a boundary point if it lies in the domain of a boundary chart but not in an interior chart. The subset of M consisting of points that are boundary points is called the *boundary of M*. The subset of M consisting of interior points in called the *interior of M*.

A manifold is called compact if it is compact as a topological space. A manifold is called *closed*, if it is compact and without a boundary.

To define a smooth structure on a manifold with boundary, we simply use the following definition of a smooth function $f : U \to \mathbf{R}^n$ where $U \subset \mathbf{H}^n$. The function f is said to be smooth if it admits a smooth extension to an open subset of $\mathbf{R}^n$. Using this definition of smoothness, we define smooth charts, atlases and structures in the usual way.

Generally results about smooth manifolds extend to similar results about smooth manifolds with boundary. A notable exception is that the result about products of manifolds. Products of manifolds with boundary are not in general manifolds with boundary.

3. Partitions of unity

Now we shall define an important tool, which is used frequently to convert local constructions on manifolds, i.e. ones that use smooth charts, to global ones. (We shall provide some concrete examples of this eventually.)

Definition 3.1. Given a function $f : M \to \mathbf{R}$, the *support* of f, is denoted as

$$supp(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

We shall need the following elementary Lemma.

Lemma 3.2. *The function*

$$f(t) = \begin{cases} e^{\frac{-1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth.

Exercise 3.3. *Prove the above Lemma.*

Definition 3.4. Consider the function

$$h(t) = \frac{f(2-t)}{(f(2-t) + f(t-1))}$$

One can verify that:

(1) h is smooth.
(2) $h(t) = 1$ if $t \leq 1$.
(3) $0 \leq h(t) \leq 1$ if $1 \leq t \leq 2$.
(4) $h(t) = 0$ if $t \geq 2$.

Using h, we construct the following function on $\mathbf{R}^n$.

$$H : \mathbf{R}^n \to [0,1] \qquad H(x) = h(|x|)$$

Note that H is an example of a type of *smooth bump function*, i.e. it satisfies:

(1) It is smooth.
(2) Its support is contained in $B_2(0)$.
(3) It maps $B_1(0)$ to 1.

Definition 3.5. (Partitions of unity) Let M be a smooth manifold and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover (usually consisting of smooth charts) of M. A *partition of unity subordinate to* $\mathcal{U}$ is a collection of smooth functions

$$\{f_\alpha\}_{\alpha \in I} \qquad f_\alpha : M \to \mathbf{R}$$

such that:

(1) $0 \leq f_\alpha(x) \leq 1$ for each $\alpha \in I, x \in M$.
(2) $supp(f_\alpha) \subset U_\alpha$.
(3) For each $x \in M$ there is a neighborhood V of x such that all but finitely many f_α vanish on V.
(4) $\sum_{\alpha \in I} f_\alpha(x) = 1$ for each $x \in M$.

Theorem 3.6. *For any open cover $\mathcal{U}$ of a smooth manifold M, there exists a partition of unity subordinate to $\mathcal{U}$.*

The theorem has a very nice and useful immediate consequence.

Corollary 3.7. *Let M be a smooth manifold. Let V be a closed set and U be an open set that contains U. Then there exists a smooth function $f : M \to \mathbf{R}$ such that $f(x) = 1$ whenever $x \in V$, and $supp(f) \subset U$.*

Proof. We take a partition of unity subordinate to the open covering $\{U, M \setminus V\}$. The function supported in U has the desired property. □

Definition 3.8. Let M be a smooth manifold and let V be a closed set. A function $f : V \to \mathbf{R}$ is said to be smooth if it admits a smooth extension $f : U \to \mathbf{R}$, for some open set U containing V.

Corollary 3.9. *Let M be a smooth manifold, and let V be a closed subset and a smooth function $f : V \to \mathbf{R}$. Then there is a smooth function $g : M \to \mathbf{R}$ such that $g \restriction V = f$.*

Proof. By definition, f admits a smooth extension to an open set U containing V. Consider a smooth bump function $h : M \to \mathbf{R}$ whose support is in an intermediate open set $V \subset U' \subset U$, and which is identically 1 on V. Then $g = fh$ is defined on U, has support in U' and obviously smoothly extends as identically 0 on $M \setminus U$. □