

ΠCAA lecture 6: quiz solutions

1) a) large spectral gap: the graph is well connected (close to a complete graph, actually)

b1) small spectral gap: the chain is close to periodic
 \Rightarrow an eigenvalue is close to -1

b2) small spectral gap: the chain is close to disconnected;
long stays in states 1 & 3 \Rightarrow an eigenvalue close to $+1$

c) small spectral gap: the chain is (very) close to periodic \Rightarrow an eigenvalue close to -1

d1) small spectral gap: the chain is close to disconnected and close to periodic \Rightarrow an eigenvalue is close to -1 and another is close to $+1$

d2) the chain is close to periodic in this case \Rightarrow small spectral gap also

2) Adding self-loops in cases b1), c) & d2) increases the spectral gap.

3) a) Assume $\exists j_0 \in S$ with $c_{j_0} < c_i$:

$$\text{Then } \sum_{j \in S} p_j c_j = \sum_{j \in S, j \neq j_0} p_j c_j + p_{j_0} c_{j_0}$$

$$< \sum_{j \in S, j \neq j_0} p_j \cdot c_i + \underbrace{p_{j_0}}_{>0} \cdot c_i = \left(\underbrace{\sum_{j \in S} p_j}_{=1} \right) \cdot c_i = c_i$$

which contradicts the assumption.

b) Assume $\exists A \subset S, A \neq \emptyset$ with $d_j = \begin{cases} -d_i & \text{if } j \in A \\ d_i & \text{if } j \notin A \end{cases}$

$$\text{Then } \sum_{j \in S} p_j d_j = d_i \left(\underbrace{-\sum_{j \in A} p_j + \sum_{j \notin A} p_j}_{< 1 \text{ in absolute value (all } p_j > 0)} \right) \neq d_i$$

c) Same story: consider $A \subset S$, $d_j = \begin{cases} -d_i & j \in A \\ d_i & j \notin A \end{cases}$

$$\sum_{j \in S} p_j d_j = d_i \left(\underbrace{-\sum_{j \in A} p_j + \sum_{j \notin A} p_j}_{< 1 \text{ in absolute value (all } p_j > 0)} \right) \neq -d_j \quad \text{and } A = \emptyset \text{ does not work either}$$