LECTURE 2

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1. Partitions of unity

Now we shall define an important tool, which is used frequently to convert local constructions on manifolds, i.e. ones that use smooth charts, to global ones. (We shall provide some concrete examples of this eventually.)

Definition 1.1. Given a function $f : M \to \mathbf{R}$, the *support* of f, is denoted as

$$supp(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

We shall need the following elementary Lemma.

Lemma 1.2. *The function*

$$f(t) = \begin{cases} e^{\frac{-1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth.

Exercise 1.3. *Prove the above Lemma.*

Definition 1.4. Consider the function

$$h(t) = \frac{f(2-t)}{(f(2-t) + f(t-1))}$$

One can verify that:
(1) h is smooth.
(2)
$$h(t) = 1 \text{ if } t \leq 1 \qquad 0 \leq h(t) \leq 1 \text{ if } 1 \leq t \leq 2 \qquad h(t) = 0 \text{ if } t \geq 2$$

Using h, we construct the following function on $\mathbf{R}^n$.

$$H : \mathbf{R}^n \to [0,1] \qquad H(x) = h(|x|)$$

Note that H is an example of a type of *smooth bump function*, i.e. it satisfies:
(1) It is smooth.
(2) Its support is contained in $B_2(0)$.
(3) It maps $B_1(0)$ to 1.

Definition 1.5. (Partitions of unity) Let M be a smooth manifold and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover (usually consisting of smooth charts) of M. A *partition of unity subordinate to* $\mathcal{U}$ is a collection of smooth functions

$$\{f_\alpha\}_{\alpha \in I} \qquad f_\alpha : M \to \mathbf{R}$$

such that:
(1) $0 \leq f_\alpha(x) \leq 1$ for each $\alpha \in I, x \in M$.
(2) $supp(f_\alpha) \subset U_\alpha$.
(3) For each $x \in M$ there is a neighborhood V of x such that all but finitely many f_α vanish on V.
(4) $\sum_{\alpha \in I} f_\alpha(x) = 1$ for each $x \in M$.

Theorem 1.6. *For any open cover $\mathcal{U}$ of a smooth manifold M, there exists a partition of unity subordinate to $\mathcal{U}$.*

The theorem has a very nice and useful immediate consequence.

Corollary 1.7. *Let M be a smooth manifold. Let V be a closed set and U be an open set that contains U. Then there exists a smooth function $f : M \to \mathbf{R}$ such that $f(x) = 1$ whenever $x \in V$, and $supp(f) \subset U$.*

Proof. We take a partition of unity subordinate to the open covering $\{U, M \setminus V\}$. The function supported in U has the desired property. □

Definition 1.8. Let M be a smooth manifold and let V be a closed set. A function $f : V \to \mathbf{R}$ is said to be smooth if it admits a smooth extension $f : U \to \mathbf{R}$, for some open set U containing V.

Corollary 1.9. *Let M be a smooth manifold, and let V be a closed subset and a smooth function $f : V \to \mathbf{R}$. Then there is a smooth function $g : M \to \mathbf{R}$ such that $g \restriction V = f$.*

Proof. By definition, f admits a smooth extension to an open set U containing V. Consider a smooth bump function $h : M \to \mathbf{R}$ whose support is in an intermediate open set $V \subset U' \subset U$, and which is identically 1 on V. Then $g = fh$ is defined on U, has support in U' and obviously smoothly extends as identically 0 on $M \setminus U$. $\square$

2. Proof of the main theorem

We shall now prove the main theorem.

Theorem 2.1. *For any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a smooth manifold M, there exists a partition of unity subordinate to $\mathcal{U}$.*

To prove this theorem, we shall need the following result. Recall that a manifold M is paracompact, if every open cover $\mathcal{U}$ admits a *open, locally finite, refinement*, which is another cover $\mathcal{V}$ with the following features:

(1) $\mathcal{V}$ is an open cover.
(2) $\mathcal{V}$ is locally finite: for each $x \in M$, there are only finitely many elements of $\mathcal{V}$ that contain x.
(3) $\mathcal{V}$ is a refinement of $\mathcal{U}$: for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $V \subset U$.

Theorem 2.2. *Every topological manifold is paracompact. In fact, given a topological manifold M, and an open cover $\mathcal{U}$ of M, and a basis of regular coordinate balls, there exists a countable, locally finite open refinement of $\mathcal{M}$ consisting of regular coordinate balls.*

Proof. Let M,$\mathcal{U}$ be as in the statement. Recall from the exercises that the manifold admits an exhaustion by compact sets, $(K_j)_{j \in \mathbf{N}}$. Let

$$V_j = K_{j+1} \setminus Int(K_j) \qquad W_j = Int(K_{j+2}) \setminus K_{j-1}$$

Then V_j is a compact set contained in the open set W_j. (See exercise below for the proof of the claim that K_{j-1} is closed, hence W_j is open.)

For each $x \in V_j$, there is some $U_x \in \mathcal{U}$ containing x, and a regular coordinate ball B_x such that

$$x \in B_x \subset U_x \cap W_j$$

The collection

$$\{B_x \mid x \in V_j\}$$

is an open cover of V_j and hence has a finite subcover which we denote as $\mathcal{Y}_j$. The union

$$\mathcal{Y} = \bigcup_{i \in \mathbf{N}} \mathcal{Y}_j$$

is a countable open cover of M. Note that since $\mathcal{Y}_j$ consists of finitely many subsets of W_j, and sets in W_j are disjoint from sets in $W_{j'}$ whenever $|j - j'| \geq 4$, we know that this union of finite subcovers is locally finite. $\square$

Exercise 2.3. *Let X be a compact subset of a Hausdorff topological space. Show that X is closed.*

Proof of Theorem 2.1. We shall prove this theorem in the case when M is a smooth manifold without boundary. We are given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a smooth manifold M, and we wish to find a partition of unity subordinate to $\mathcal{U}$.

By theorem 2.2, there is a locally finite, countable, open refinement

$$\mathcal{X} = \{B_1, B_2, ..., B_n, ...\}$$

of $\mathcal{U}$ consisting of regular coordinate balls. By the exercise below, it follows that the set $\{\overline{B_1}, \overline{B_2}, ...\}$ is also locally finite.

Since each B_i is a regular coordinate ball in some U_α, there is a coordinate ball B_i' such that $\overline{B_i} \subset B_i'$ and there is a smooth coordinate map $\phi_i : B_i' \to \mathbf{R}^n$ such that

$$\phi_i(\overline{B_i}) = \overline{B_{r_i}(0)} \qquad \phi_i(B_i') = B_{r_i'}(0) \qquad \text{for some positive reals } r_i < r_i'$$

For each $i \in \mathbf{N}$, we define a function $f_i : M \to \mathbf{R}$ as

$$f_i = H_i \circ \phi_i \text{ on } B_i' \qquad f_i = 0 \text{ on } M \setminus \overline{B_i}$$

where H_i is the smooth bump function from 1.4 that has been "rescaled" (check how this is done!). The functions f_i are well defined and smooth (check this!) and $supp(f_i) = \overline{B_i}$.

Define

$$f : M \to \mathbf{R} \qquad f(x) = \sum_{i \in \mathbf{N}} f_i(x)$$

We check that f is smooth because of the local finiteness condition and that $f(x) > 0$ for each $x \in M$. We define the functions

$$g_i : M \to \mathbf{R} \qquad g_i(x) = \frac{f_i(x)}{f(x)}$$

Clearly, g_i are smooth functions that satisfy:

(1) $0 \leq g_i(x) \leq 1$ for each $x \in M$.
(2) $\sum_{i \in \mathbf{N}} g_i(x) = 1$ for each $x \in M$.

This provides a partition of unity for the open cover $\mathcal{X} = \{B_1, B_2, ...B_n, ...\}$. However, note that we still need to find a partition of unity for $\mathcal{U}$.

For each $i \in \mathbf{N}$, we choose a number $\zeta(i) \in I$ such that $B_i \subseteq U_{\zeta(i)}$. Note that such a choice is possible since $\mathcal{X} = \{B_1, B_2, ...B_n, ...\}$ is a refinement of $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. The set of these choices defines a function $\zeta : \mathbf{N} \to I$.

For each $\alpha \in I$, we define

$$\psi_\alpha : M \to \mathbf{R} \qquad \psi_\alpha = \sum_{\{i \in \mathbf{N} | \zeta(i) = \alpha\}} g_i$$

It follows that $supp(\psi_\alpha) \subseteq U_\alpha$ and that $\{\psi_\alpha \mid \alpha \in I\}$ is the desired partition of unity for $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

□

Exercise 2.4. *Let $\{B_1, B_2, ...B_n, ...\}$ be a locally finite collection of subsets of a topological space. Show that $\{\overline{B_1}, \overline{B_2}, ...\}$ is also locally finite.*