LECTURE 4

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1. Review of smooth maps and diffeomorphisms in Euclidean space

Let $U \subset \mathbf{R}^n$ be an open subset and let $f : U \to \mathbf{R}$ be a real valued function. The recall that

$$\frac{\partial f}{\partial x^j}(a) = \lim_{t\to 0} \frac{f(a+te^j) - f(a)}{t}$$

More generally, for a vector valued function

$$F : U \to \mathbf{R}^m \qquad F = (F^1, ..., F^m)$$

we define $\frac{\partial F^i}{\partial x^j}$. The matrix $A_{i,j} = (\frac{\partial F^i}{\partial x^j})$ is called the *Jacobian matrix* and its determinant is called the *Jacobian determinant*. The function F is said to be of class C^k, if the k-th order derivatives exist and are continuous. The function is said to be *smooth*, if it is infinitely differentiable. It is called a *diffeomorphism*, if it is smooth and bijective, and the inverse function is also smooth.

2. Geometric tangent vectors

Given a point $a \in \mathbf{R}^n$, a *geometric tangent vector* at a is the set of pairs

$$\mathbf{R}^n_a = \{(a, v) \mid v \in \mathbf{R}^n\}$$

The pair (a, v) is usually denoted as v_a and the space $\mathbf{R}^n_a$ is endowed with the natural structure of a vector space. Recall that $C^\infty(\mathbf{R}^n)$ is the set of smooth functions from $\mathbf{R}^n$ to $\mathbf{R}$. This is naturally an infinite dimensional vector space with pointwise addition and scalar multiplication

$$(f + g)(x) = f(x) + g(x) \qquad (cf)(x) = c(f(x))$$

Any geometric tangent vector v_a yields a map

$$D_v \mid_a : C^\infty(\mathbf{R}^n) \to \mathbf{R}$$

which takes the directional derivative

$$D_v \mid_a (f) = D_v(f(a)) = \frac{d}{dt} \mid_{t=0} f(a + tv)$$

This is linear over $\mathbf{R}$, i.e.

$$D_v \mid_a (cf + dg) = cD_v \mid_a (f) + dD_v \mid_a (g) \qquad f, g \in C^\infty(\mathbf{R}^n), c, d \in \mathbf{R}$$

and it satisfies the product rule

$$D_v \mid_a (fg) = f(a)D_v \mid_a g + g(a)D_v \mid_a f$$

Such a derivation can be written concretely as:

$$D_v \mid_a f = v^i \frac{\partial f}{\partial x^i}(a) \qquad v = v^i E_i$$

With this construction in mind, we can "abstract" this as follows. Given $a \in \mathbf{R}^n$, a map

$$w : C^\infty(\mathbf{R}^n) \to \mathbf{R}$$

is called a *derivation at a* if it is linear over $\mathbf{R}$ and satisfies the product rule, i.e.

$$w(fg) = f(a)wg + g(a)wf$$

We denote by $T_a\mathbf{R}^n$ as the set of all derivations at a. This is clearly a real vector space under the operation

$$(w_1 + w_2)f = w_1 f + w_2 f \qquad (cw)f = c(wf)$$

Exercise 2.1. *Check the above, i.e. indeed the set of all derivations form a vector space.*

We will show that $T_a\mathbf{R}^n$ is finite dimensional and naturally isomorphic to $\mathbf{R}^n$.

Lemma 2.2. *Suppose $a \in \mathbf{R}^n, w \in T_a\mathbf{R}^n$ and $f, g \in C^\infty(\mathbf{R}^n)$. Then*

(1) *If f is a constant function, then $wf = 0$.*
(2) *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Exercise 2.3. *Prove the above Lemma using linearity and the product rule.*

Proposition 2.4. *Let $a \in \mathbf{R}^n$.*

(1) *For each geometric tangent vector $v_a \in \mathbf{R}^n$, the map*

$$D_v \mid_a : C^\infty(\mathbf{R}^n) \to \mathbf{R}$$

is a derivation at a.

(2) *The map $v \to D_v \mid_a$ is an isomorphism from $\mathbf{R}^n \to T_a\mathbf{R}^n$.*

Proof. Part a is immediate from the definitions. To prove that the map in part b is an isomorphism, first note that it is clearly linear. We will show that it is injective and surjective.

Suppose v_a has the property that $D_v \mid_a$ is the zero derivation. Let f be the j-th coordinate function, which is clearly smooth. Then

$$D_v \mid_a f = v^j \qquad v = v^i E_i$$

This is zero for each coordinate function if and only if v is the zero vector. This proves that the map is injective.

Now we shall prove surjectivity. Let $w \in T_a\mathbf{R}^n$ be a derivation. Let $v^i = w(x^i)$, where x^i is the i'th coordinate function. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Using Taylor's theorem we can write

$$f(x) = f(a) + \sum_{1 \le i \le n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{1 \le i,j \le n} (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))dt$$

Using the product rule, when we take the derivation wf, the first and the third term above vanishes and we obtain

$$wf = \sum_{1 \le i \le n} w(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)) = \sum_{1 \le i \le n} \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i))$$

$$= \sum_{1 \le i \le n} \frac{\partial f}{\partial x^i}(a) v^i = D_v \mid_a f \qquad v = v^i E_i$$

□

Corollary 2.5. *For any $a \in \mathbf{R}^n$, the derivations*

$$\frac{\partial}{\partial x^1} \mid_a, \ldots, \frac{\partial}{\partial x^n} \mid_a$$

form a basis for $T_n\mathbf{R}^n$.

3. Tangent vectors on manifolds

Let M be a smooth manifold of dimension n. The vector space of smooth functions from $M \to \mathbf{R}$ is denoted as $C^\infty(M)$. A linear map $w : C^\infty(M) \to \mathbf{R}$ is called a *derivation at p* if is satisfies the product rule:

$$w(fg) = f(p)w(g) + g(p)w(f)$$

The set of all derivations at $p \in M$ is denoted as $T_p(M)$, and an element $v \in T_p(M)$ is called a *tangent vector atp.*

The following is an analogue of the Lemma from the previous section.

Lemma 3.1. *Suppose M is a smooth manifold, $p \in M, v \in T_p(M), f, g \in C^\infty(M)$. Then the following hold*

(1) *If f is a constant function, then $v(f) = 0$.*
(2) *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Using the Theorem from the previous section, we conclude that for each $p \in M$, $T_p(M)$ is an n-dimensional real vector space.

3.1. **The differential of a smooth map.** Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. For each $p \in M$, we define a map

$$dF_p : T_p(M) \to T_{F(p))}(M)$$

as follows. Let $v \in T_p(M)$. Then $dF_p(v) = w$ where w is the derivation

$$w(f) = v(f \circ F)$$

Note that the above makes sense since $f \circ F$ is also a smooth function. To check that this is a derivation, we need to check that it is linear and that it satisfies the product rule. Linearity is immediate from the definition, and to check the product rule:

$$w(fg) = v(fg \circ F) = v((f \circ F)(g \circ F)) = (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) = f(F(p))w(g) + g(F(p))w(f)$$

We now summarise some basic properties of this differential operator.

Proposition 3.2. *Let M, N, P be smooth manifolds and let*

$$F : M \to N \qquad G : N \to P$$

be smooth maps, and let $p \in M, q = F(p), r = G(F(p))$. Then the following holds.

1. *$dF_p : T_p(M) \to T_q(N)$ is linear.*
2. *$d(G \circ F)_p = dG_q \circ dF_p : T_p(M) \to T_r(P)$.*
3. *If F is a diffeomorphism, then $dF_p : T_p(M) \to T_q(N)$ is an isomorphism and $(dF_p)^{-1} = dF_p^{-1}$.*

The next proposition demonstrates that the derivations provided by tangent vectors are "local".

Proposition 3.3. *Let M be a smooth manifold, $p \in M$ and $v \in T_p(M)$. If $f, g \in C^\infty(M)$ agree on some neighbourhood of p then $v(f) = v(g)$.*

Proof. Let $h = f - g$. Clearly, h vanishes on a neighbourhood of p as f, g agree on such a neighbourhood. Let ϕ be a smooth bump function that is identically 1 on $supp(h)$ and whose support satisfies

$$supp(\phi) \subset M \setminus \{p\}$$

Then $v(\phi h) = 0$ thanks to Lemma 3.1. Note that ϕh is indeed equal to h, by definition. It follows that $v(h) = 0$ and so by linearity $v(f - g) = 0 \implies v(f) = v(g)$. □

Proposition 3.4. *Let M be a smooth manifold and let $U \subseteq M$ be an open set with the inclusion map $i : U \to M$. For every $p \in M$, the differential $di : T_p(U) \to T_p(M)$ is an isomorphism.*

Proof. First we show that the differential is injective. Suppose $v \in T_p(U)$ lies in the kernel. Let B be an open ball around p such that $\overline{B} \subset U$. Let $f \in C^\infty(U)$. The extension theorem (from the exercises!) guarantees that there exists a smooth function $\tilde{f} : M \to \mathbf{R}$ such that $\tilde{f} \restriction \overline{B} = f \restriction \overline{B}$. Proposition 3.3 implies that

$$v(f) = v(\tilde{f} \restriction_U) = v(\tilde{f} \circ i) = di_p(v)(\tilde{f}) = 0$$

However, $v(f) = 0$ cannot hold for each $f \in C^\infty(U)$ since $v \neq 0$. This is a contradiction. Hence the differential is injective.

Now we shall prove surjectivity. Suppose that $w \in T_p(M)$ is arbitrary. Let $\overline{B}$ be as above. Define an operator $v : C^\infty(U) \to \mathbf{R}$ by $v(f) = w(\tilde{f})$ where $\tilde{f}$ is any smooth function in $C^\infty(M)$ that agrees with f on $\overline{B}$. Note that by Proposition 3.3, the definition does not depend on the choice of $\tilde{f}$ and hence is well defined and is obviously a derivation at p. For any $g \in C^\infty(M)$ it follows that

$$di_p(v)(g) = v(g \circ i) = w(\widetilde{g \circ i}) = w(g)$$

so $v \mapsto w$. □

Now we prove the following fundamental statement about the dimension of the tangent space.

Proposition 3.5. *If M is an n-dimensional smooth manifold and $p \in M$, the tangent space $T_p(M)$ is an n-dimensional vector space.*

Proof. Gievn $p \in M$, let (U, ϕ) be a smooth coordinate chart containing p. Because ϕ is a diffeomorphism from U to an open subset U' of $\mathbf{R}^n$, It follows that $T_p(U)$ is isomorphic to $T_{\phi(p)}(U')$ by the last part of Proposition 3.2. Since $T_p(U)$ is isomorphic to $T_p(M)$ and $T_p(U')$ is isomorphic to $T_p(\mathbf{R}^n)$ by Proposition 3.4, we are done. □

4. The tangent bundle

It is often useful to consider all tangent vectors of all points on a smooth manifold M as a single object. This is called the *tangent bundle*, and is defined as the set

$$TM = \sqcup_{p\in M} T_p(M)$$

Elements of the tangent bundle as denoted as pairs (p, v) where $p \in M$ and $v \in T_p(M)$. The natural projection map $TM \to M$ is simply $(p, v) \mapsto p$. The tangent bundle is more than simply a union of vector spaces, indeed it has the structure of a smooth manifold. We shall prove this (hopefully!) eventually in the course.

Proposition 4.1. *For any smooth manifold M of dimension n, the tangent bundle TM has a natural topology and a smooth structure that make it a smooth manifold of dimension $2n$. With respect to this smooth structure, the natural projection map $TM \to M$ is smooth.*