ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 11 Principles of Digital Communications

Homework 5 Oct. 13, 2020

PROBLEM 1. Assume $\{X_n\}_{-\infty}^{\infty}$ and $\{Y_n\}_{-\infty}^{\infty}$ are two i.i.d. processes (individually) with the same alphabet, with the same entropy rate $H(X_0) = H(Y_0) = 1$ and independent from each other. We construct two processes Z and W as follows:

- To construct the process Z, we flip a fair coin and depending on the result $\Theta \in \{0, 1\}$ we select one of the processes. In other words, $Z_n = \Theta X_n + (1 - \Theta)Y_n$.
- To construct the process W, we do the coin flip at every time n. In other words, at every time n we flip a coin and depending on the result $\Theta_n \in \{0, 1\}$ we select X_n or Y_n as follows $W_n = \Theta_n X_n + (1 - \Theta_n)Y_n$.

(a) Are Z and W stationary processes? Are they i.i.d. processes?

(b) Find the entropy rate of Z and W. How do they compare? When are they equal?

Recall that the entropy rate of the process U (if exists) is $\lim_{n\to\infty} \frac{1}{n} H(U_1, \cdots, U_n)$.

PROBLEM 2. Let the alphabet be $\mathcal{X} = \{a, b\}$. Consider the infinite sequence $X_1^{\infty} = ababababababab\ldots$

(a) What is the compressibility of $\rho(X_1^{\infty})$ using finite-state machines (FSM) as defined in class? Justify your answer.

(b) Design a specific FSM, call it M, with at most 4 states and as low a $\rho_{\mathrm{M}}(X_1^{\infty})$ as possible. What compressibility do you get?

(c) Using only the result in point (a) but no specific calculations, what is the compressibility of X_1^{∞} under the Lempel–Ziv algorithm, i.e., what is $\rho_{\mathrm{LZ}}(X_1^{\infty})$?

(d) Re-derive your result from point (c) but this time by means of an explicit computation.

PROBLEM 3. We have shown in class that

$$\binom{n}{k} \leq 2^{n \mathrm{h}_2\left(\frac{k}{n}\right)}.$$

(a) Given $n \in \mathbb{N}_+$ and $n_1, n_2, \ldots, n_K \in \mathbb{N}$ such that $\sum_{i=1}^{n} n_i = n$, we define the quantity $\binom{n}{n_1 n_2 \ldots n_K} = \frac{n!}{n_1! n_2! \ldots n_K!}$.
Show that

$$\binom{n}{n_1 n_2 \ldots n_K} \leq 2^{n \mathrm{h}(p_1, p_2, \ldots, p_K)},$$

where $p_i = \frac{n_i}{n}$ and $\mathrm{h}(p_1, \ldots, p_K) = -\sum_{i=1}^{K} p_i \log(p_i)$.

Let $U_1, U_2, \ldots$ be the letters generated by a memoryless source with alphabet $\mathcal{U} = \{u_1, u_2, \ldots, u_K\}$, i.e., $U_1, U_2, \ldots$ are i.i.d. random variables taking values in the alphabet $\mathcal{U}$ according to the distribution $q = \{q_1, q_2, \ldots, q_K\}$.

(b) We want to compress this source without any idea about its distribution. Describe an optimal universal code that achieves this goal. Give a proof of its optimality. Hint: Use the same idea as for the binary source case.

(c) What if the source is not i.i.d. Will your code still be optimal?

Problem 4. Consider the discrete memoryless channel $Y = X + Z \pmod{11}$, where

$$\Pr(Z = 1) = \Pr(Z = 2) = \Pr(Z = 3) = 1/3$$

and $X \in \{0, 1, \dots, 10\}$. Assume that Z is independent of X.

(a) Find the capacity.

(b) What is the maximizing $p^*(x)$?

Problem 5. Suppose there are two discrete memoryless channels which are characterized by $(\mathcal{X}_1, p(x_1|y_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(x_2|y_2), \mathcal{Y}_2)$ respectively. Assume further that $\mathcal{Y}_1$,$\mathcal{Y}_2$ and $\mathcal{X}_1$,$\mathcal{X}_2$ are disjoint (i.e. $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$ and $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$). Find the capacity C of the union of these two channels in terms of individual capacities C_1 and C_2. A union of these two channels means that the user can send one bit at a time using only one of these channels.

(Hint: You can flip a coin with optimal probability distribution to determine which channel to use.)