LECTURE 5

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1. Local coordinates

Let M be an n-dimensional manifold and let $p \in M$. Let (U, ϕ) be a chart containing p and let $\phi(p) = q$. Recall the isomorphisms

$$d\phi_p : T_p(U) \to T_q(\mathbf{R}^n) \qquad d\phi_q^{-1} : T_q(\mathbf{R}^n) \to T_p(U)$$

Since ϕ is a diffeomorphism, the tangent vectors

$$d\phi_q^{-1} \frac{\partial}{\partial x^i} |_q = \frac{\partial}{\partial \phi^i} |_p$$

for a basis for $T_p(M)$ and are called the coordinate vectors at p associated to the chart (U, ϕ).

The action of $\frac{\partial}{\partial \phi^i} |_p$ on $f \in C^\infty(M)$ is defined as

$$\frac{\partial}{\partial \phi^i} |_p (f) = \frac{\partial}{\partial x^i} |_q (f \circ \phi^{-1})$$

In particular, note that

$$\frac{\partial}{\partial \phi^i} |_p \phi^j = 1 \text{ if } j = i \text{ and } 0 \text{ otherwise}$$

To make the notation simpler, sometimes we may also refer to $d\phi_p$ above as simply $(\phi_p)_*$. We shall do this for the rest of the lecture. It is important to get used to both forms of notation since the literature is quite mixed.

Let M, N be smooth manifolds. Let $F : M \to N$ be a smooth map, $p \in M$ and $F(p) = q \in N$. Let (U, ϕ) be a chart containing p and (V, ψ) be a chart containing q. We have the corresponding bases

$$(\frac{\partial}{\partial \phi^i} |_p) \qquad (\frac{\partial}{\partial \psi^i} |_q)$$

of $T_p(M), T_q(N)$ respectively. Then

$$F_* \frac{\partial}{\partial \phi^i} |_p = \sum_{1 \leq j \leq n} (\frac{\partial}{\partial \phi^i} |_p (\psi^j \circ F)) \frac{\partial}{\partial \psi^j} |_q$$

These coefficients are the coefficients of the Jacobian matrix of the coordinate representation

$$\hat{F} = \psi \circ F \circ \phi^{-1}$$

given as

$$\frac{\partial}{\partial \phi^i} |_p (\psi^j \circ F) = \frac{\partial}{\partial x^i} |_p (\psi^j \circ F \circ \phi^{-1}) = \frac{\partial \psi^j \circ F \circ \phi^{-1}}{\partial x^i} (\phi(p)) = (D\hat{F}(\phi(p))$$

For the special case when $M = N$ and $F = id$ (the identity map), we obtain

$$\frac{\partial}{\partial \phi^i} |_p = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial \phi^i} (\psi^j)) \frac{\partial}{\partial \psi^j} |_p$$

Example 1.1. (Polar coordinates) Let $W = \mathbf{R}_{>0} \times (0, 2\pi)$. The map

$$\phi : W \to \mathbf{R}^2 \qquad (r, \theta) \to (r\cos(\theta), r\sin(\theta))$$

is a diffeomorphism onto its image. The inverse, (U, ϕ^{-1}) is a smooth chart for $\mathbf{R}^2$, called *polar coordinates*. The coordinate functions of ϕ^{-1} are written as (r, θ) and the standard coordinates are simply (x, y).

Let

$$p = (x_0, y_0) \qquad q = \phi^{-1}(p) = (r_0, \phi_0)$$

Then

$$\frac{\partial}{\partial r} |_p = (\frac{\partial x}{\partial r}(q)) \frac{\partial}{\partial x} |_p + (\frac{\partial y}{\partial r}(q)) \frac{\partial}{\partial y} |_p$$

$$\frac{\partial}{\partial \theta} \mid_p = (\frac{\partial x}{\partial \theta}(q))\frac{\partial}{\partial x} \mid_p + (\frac{\partial y}{\partial \theta}(q))\frac{\partial}{\partial y} \mid_p$$

Exercise 1.2. *Finish the above computation.*

2. Definition of tangent space by means of smooth curves

Let M be a smooth manifold. A *smooth curve* is a smooth map $\gamma : J \to M$ where J is an open subset of $\mathbf{R}$. The tangent vector to γ at $r \in J, s = \gamma(r)$ is the "push-forward" vector

$$\gamma'(r) = \gamma_* \frac{d}{dt} \mid_r \in T_s(M)$$

where $\frac{d}{dt}$ is the standard basis vector of the vector space $T_r(\mathbf{R})$.

Given a smooth function $f : M \to \mathbf{R}$, we have

$$\gamma'(t)(f) = \frac{d}{dt} \mid_r (f \circ \gamma)$$

In terms of coordinate vectors in terms of a given chart (U, ϕ), we can express $\gamma'(t)$ as

$$\gamma'(t) = \sum_{1 \leq i \leq n} (\gamma^i)'(t) \frac{\partial}{\partial \phi^i} \mid_s \qquad \gamma^i = \phi^i \circ \gamma$$

and $(\gamma^i)'(t)$ is simply the ordinary differentiation.

Exercise 2.1. *Show that every tangent vector in $T_p(M)$ is the tangent vector to a smooth curve.*

Example 2.2. Consider the smooth curve

$$\gamma : \mathbf{R} \to \mathbf{R}^2 \qquad \gamma(t) = (\cos(t), \sin(t))$$

Then

$$\gamma'(t_0) = -\sin(t_0)\frac{\partial}{\partial x} \mid_{s_0} + \cos(t_0)\frac{\partial}{\partial y} \mid_{s_0} \qquad s_0 = (\cos(r_0), \sin(r_0))$$

Proposition 2.3. *Let $F : M \to N$ be a smooth map between smooth manifolds. Let $\gamma : J \to M$ be a smooth curve where $J \subset \mathbf{R}$ is an open interval. Then*

$$F_*(\gamma'(t)) = (F \circ \gamma)'(t)$$

Proof. This follows from the definitions. □

3. Vector bundles

Let M be a smooth manifold. A *smooth vector bundle of rank k over M* is a smooth manifold E together with a surjective map

$$\pi : E \to M$$

such that:

(1) For each $p \in M$, the set $\pi^{-1}(p) = E_p$ is a k-dimensional vector space, called the *fibre of E over p*.
(2) For each $p \in M$, there exists a neighbourhood U of p and a diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbf{R}^k$$

called a *local trivialisation of E over U* such that

$$\phi(E_p) = \{p\} \times \mathbf{R}^k$$

and

$$\phi \mid_{E_p} : E_p \to \{p\} \times \mathbf{R}^k$$

is a vector space isomorphism.

We call E the total space of the bundle and M its base.

Given vector bundles $(E, \pi_1), (E', \pi_2)$ over the same manifold M, a bundle map $F : E \to E'$ is a smooth map satisfying that:

(1) $\pi_2 \circ F = \pi_1$.
(2) The map $F \mid_{E_p} : E_p \to E'_p$ is linear.

A bundle isomorphism is a bundle map whose inverse is also a bundle map.

A basic example of a bundle over M is the trivial bundle, which is simply $M \times \mathbf{R}^k$.

4. The tangent bundle as a smooth vector bundle

Our goal now will be to show that the tangent bundle is a smooth vector bundle.

Lemma 4.1. *Let M be a smooth manifold of dimension n. The tangent bundle TM has a natural topology and smooth structure, making it a smooth $2n$-dimensional bundle with the natural projection map $\pi : TM \to M$.*

Proof. We shall define the smooth charts for TM. Let (U, ϕ) be a smooth chart for M. Then we define the map

$$\Phi : \pi^{-1}(U) \to \mathbf{R}^{2n} \qquad \Phi(v^i \frac{\partial}{\partial \phi^i} \mid_p) \to (\phi(p), v) \qquad v = (v^i E_i)$$

We define the following basis of the topology: the open sets in this basis are the inverse images of open balls in $\mathbf{R}^{2n}$ under the maps Φ as above. So by construction, each such map Φ is a homeomorphism onto its image.

□