LECTURE 5

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1. The tangent bundle as a smooth vector bundle

Our goal now will be to show that the tangent bundle is a smooth vector bundle.

Proposition 1.1. *Let M be a smooth manifold of dimension n. The tangent bundle TM has a natural topology and smooth structure, making it a smooth $2n$-dimensional bundle with the natural projection map $\pi : TM \to M$. This makes it a smooth vector bundle of rank n.*

Proof. We shall define the smooth charts for TM. Let (U, ϕ) be a smooth chart for M. Then we define the map

$$\Phi : \pi^{-1}(U) \to \mathbf{R}^{2n} \qquad \Phi(v^i \frac{\partial}{\partial \phi^i} \restriction_p) \to (\phi(p), v) \qquad v = (v^i E_i)$$

We define the following basis of the topology: the open sets in this basis are the inverse images of open balls in $\mathbf{R}^{2n}$ under the maps Φ as above. So by construction, each such map Φ is a homeomorphism onto its image. It is easy to check that this is a basis and that the resulting topology is second countable and Hausdorff.

So for the smooth chart (U, ϕ) for M we have a smooth chart $(\pi^{-1}(U), \Phi)$ defined as above. Let (V, ψ) be another smooth chart for M and let $(\pi^{-1}(V), \Psi)$ be the corresponding smooth chart for TM. Let

$$\eta = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

be the transition map for the charts on the manifold. The transition map for the charts $(\pi^{-1}(U), \Phi), (\pi^{-1}(V), \Psi)$ is given by

$$\Psi \circ \Phi^{-1} : \phi(U \cap V) \times \mathbf{R}^n \to \psi(U \cap V) \times \mathbf{R}^n \qquad (x, v) \mapsto (\psi \circ \phi^{-1}(x), \eta_*(v))$$

which is clearly smooth.

To check that this is a vector bundle of rank n, note that each fibre $\pi^{-1}(p)$ for $p \in M$ is the vector space $T_p(M)$. Indeed, for each chart (U, ϕ) containing p, $\pi^{-1}(U)$ is homeomorphic to $U \times \mathbf{R}^n$. This provides the *local trivialisation.*

□

Definition 1.2. Let (E, π) be a smooth vector bundle of rank k over a smooth manifold M. A section is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$. A *smooth section* is a section that is a smooth map (between manifolds M, E).

If $U \subset M$ is an open set, then we define $E \restriction_U = \pi^{-1}(U)$. Note that $E \restriction_U$ is also a vector bundle of the same rank as E. A smooth section of $E \restriction_U$ is called *a smooth section of E over U.*

The set of all sections (over E or U) is endowed with the operation of pointwise addition, i.e.:

$$(\sigma_1 + \sigma_2)(p) = \sigma_1(p) + \sigma_2(p)$$

and multiplication by $f \in C^\infty(M)$, i.e.:

$$(f\sigma)(p) = f(p)\sigma(p)$$

This makes the set of all smooth sections of E (over E or U) a module over the ring $C^\infty(U)$.

A *local frame for E over U* is an ordered k-tuple $(\sigma_1, ..., \sigma_k)$ where each σ_i is a smooth section of E over U such that $(\sigma_1(p), ..., \sigma_k(p))$ is a basis for the fibre E_p for each $p \in U$. Is it called a *global frame* if $U = M$.

Proposition 1.3. *A smooth vector bundle is trivial iff it has a global frame.*

Example 1.4. (Mobius band) This is perhaps the easiest example of a non-trivial bundle. The Mobius band E is the quotient space of $[0, 1] \times \mathbf{R}$ given by the equivalence relation $(0, y) \sim (1, -y)$.

We first define the smooth structure on E by means of a system of charts. Let

$$V_1 = \{[(x, t)] \mid x \in (0, 1), t \in \mathbf{R}\} \qquad V_2 = \{[(x, t)] \mid x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], t \in \mathbf{R}\}$$

We have charts

$$\phi_1 : V_1 \to (0, 1) \times \mathbf{R} \qquad [(x, t)] \to (x, t)$$

$$\phi_2 : V_2 \to (-\frac{1}{2}, \frac{1}{2}) \times \mathbf{R} \qquad \phi_2([(x,t)]) \to (x,t) \text{ if } x \in [0, \frac{1}{2}) \qquad \phi_2([(x,t)] = (1-x, -t) \text{ if } x \in (\frac{1}{2}, 1]$$

We would like to show that E is a non-trivial smooth 1-bundle over the circle $\mathbf{S}^1$, which we view as $[0,1]/0 \sim 1$ with charts

$$U_1 = (0,1) \qquad U_2 = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]/0 \sim 1$$

$$\nu_1 : U_1 \to (0,1), [x] \to x$$

$$\nu_2 : U_2 \to (-\frac{1}{2}, \frac{1}{2}) \qquad [x] \to x \text{ if } x \in [0, \frac{1}{2}) \qquad [x] \to 1 - x \text{ if } x \in (\frac{1}{2}, 1]$$

The natural projection map is

$$\pi : E \to \mathbf{S}^1, [(x,s)] \to [x] \in \mathbf{S}^1$$

With this projection map E is a smooth vector bundle over $\mathbf{S}^1$.

We would like to show that this is a non-trivial vector bundle. That is, it is not diffeomorphic to the trivial bundle $\mathbf{S}^1 \times \mathbf{R}$. If this was the case, then by the previous proposition, we can find a global smooth frame $\sigma : \mathbf{S}^1 \to E$ (consisting of a single section, since the bundle is one dimensional). Since the function

$$\eta_1 = \phi_1 \circ \sigma \circ \nu_1^{-1} : (0,1) \to (0,1) \times \mathbf{R} \qquad \eta_1(x) = (x, t(x))$$

for some function t is smooth, hence t is smooth. Note that since σ is a global frame, x cannot change sign.

Using the transition maps, we compute

$$\eta_2 = \phi_2 \circ \sigma \circ \nu_2^{-1} : (-\frac{1}{2}, \frac{1}{2}) \to (-\frac{1}{2}, \frac{1}{2}) \times \mathbf{R}$$

restricted to $(0, \frac{1}{2}) \cup (-\frac{1}{2}, 0)$ as

$$\eta_2(x) = (x, t(x)) \text{ if } x \in (0, \frac{1}{2}) \qquad \eta_2(x) = (x, -t(x+1))$$

This means that $\eta_2(0) = (0,0)$ and hence $\sigma([0]) = 0$, which is a contradiction.

Exercise 1.5. *Fill in the details in the last paragraph of the proof, by writing down the transition functions*

$$\nu_1 \circ \nu_2^{-1} \qquad \phi_2 \circ \phi_1^{-1}$$

1.1. **Vector fields.** Let M be a smooth manifold. A *vector field* is a section of TM. A *smooth vector field* is a smooth section of TM. Given a vector field $X : M \to TM$, we usually write $X(p)$ as X_p. Addition and multiplication by $f \in C^\infty(M)$ is denoted as

$$(X+Y)_p = X_p + Y_p \qquad (fX)_p = f(p)X_p$$

Given a smooth chart (U, ϕ), we write

$$X_p = \sum_{1 \le i \le n} X^i(p) \frac{\partial}{\partial \phi^i} |_p$$

for $p \in U$ and the functions $X^i : U \to \mathbf{R}$ are the component functions.

Example 1.6. (coordinate vector fields) Given a chart (U, ϕ) the vector field $X_p = \frac{\partial}{\partial \phi^i}$ for $p \in U$ and fixed $i \in \{0, ..., n\}$.

On $T\mathbf{S}^1$ we can define a nowhere vanishing vector field (i.e. the value at each point is non zero), however, this is impossible for $T\mathbf{S}^2$ this is impossible thanks to the Hairy ball theorem.

Now given a smooth vector field X on M, we wish to find a curve whose tangent vector at each point is the vector field at that point. A smooth curve $\gamma : J \to M$ is called an *integral curve for* X if $\gamma'(t) = X_{\gamma(t)}$ for each $t \in J$. In local coordinates (U, ϕ), this boils down to solving the system of ODEs

$$(\gamma^i)'(t) = X^i(\gamma(t)) \qquad 1 \le i \le n$$

For instance, on $\mathbf{R}^n$, one has $\gamma(t) = (t, 0, ..., 0)$ as the integral curve of $\frac{\partial}{\partial x^1}$. More generally, $\gamma(t) = (t, 0, ..., 0) + y$ for fixed $y \in \mathbf{R}^n$ is also an integral curve for the same field. If we specify that $\gamma(0) = y$ for some y, then this is the unique integral curve of $\frac{\partial}{\partial x^1}$ with this property.

Example 1.7. Consider the vector field $x^2 \frac{\partial}{\partial x}$ on $\mathbf{R}$. We wish to find an integral curve with $\gamma(0) = 1$. This involves solving

$$x'(t) = (x(t))^2 \qquad x(0) = 1$$

Using the separation of variables method, the maximal solution is $x(t) = \frac{1}{1-t}$ for $x \in (-\infty, 1)$.

This shows that we may not always have a globally defined integral curve over a vector field. So we often ask for a "local existence". That is, given a point, is there a integral curve for a vector field starting at that point? The answer is yes.

Theorem 1.8. *Let X be a smooth vector field. Then for each $p \in M$ there exists a unique maximal integral curve $\gamma : J \to M$ starting at p (i.e.$\gamma(0) = p$) where $J \subset \mathbf{R}$ is an open interval containing* 0.

2. Submanifolds

Let $F : M \to N$ be a smooth map between smooth manifolds. We say that F is an *immersion* if

$$F_* : T_p(M) \to T_{F(p)}(M)$$

is injective at each point $p \in M$. We say that F is a *submersion* if

$$F_* : T_p(M) \to T_{F(p)}(M)$$

is surjective at each point $p \in M$. We say that F is a *smooth embedding* if it is an immersion that is a homeomorphism onto its image.

The rank of F at a point $p \in M$ is the rank of the linear map F_*. If the rank is constant, then we denote this as $rank(F)$.

Example 2.1. The *standard immersion* is the map

$$\mathbf{R}^n \to \mathbf{R}^m \qquad (x^1, ..., x^n) \to (x^1, ..., x^n, 0, ..., 0), m > n$$

The *standard submersion* is the map

$$\mathbf{R}^n \to \mathbf{R}^m \qquad (x^1, ..., x^n) \to (x^1, ..., x^m), m < n$$

A smooth curve $\gamma : J \to M$ is an immersion if and only if $\gamma'(t) \neq 0$ for each $t \in J$. The inclusion $\mathbf{S}^n \to \mathbf{R}^{n+1}$ is an immersion, where both manifolds are endowed with the standard smooth structures. Given smooth manifolds $M_1, ..., M_k$, the projections

$$M_1 \times ... \times M_k \to M_i$$

are submersions.

Let $J = (-\frac{\pi}{2}, \frac{3\pi}{2})$. Consider the curve

$$\gamma : J \to \mathbf{R}^2 \qquad \gamma(t) = (sin(2t), cos(t))$$

This curve is an injective immersion yet it is not an embedding.

Exercise 2.2. *Show that the curve in the last example is not an embedding.*