LECTURE 7

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1. Submanifolds

Let $F : M \to N$ be a smooth map between smooth manifolds. We say that F is an *immersion* if

$$F_* : T_p(M) \to T_{F(p)}(M)$$

is injective at each point $p \in M$. We say that F is a *submersion* if

$$F_* : T_p(M) \to T_{F(p)}(M)$$

is surjective at each point $p \in M$. We say that F is a *smooth embedding* if it is an immersion that is a homeomorphism onto its image.

The rank of F at a point $p \in M$ is the rank of the linear map F_*. If the rank is constant, then we denote this as $rank(F)$.

Example 1.1. The *standard immersion* is the map

$$\mathbf{R}^n \to \mathbf{R}^m \qquad (x^1, ..., x^n) \to (x^1, ..., x^n, 0, ..., 0), m > n$$

The *standard submersion* is the map

$$\mathbf{R}^n \to \mathbf{R}^m \qquad (x^1, ..., x^n) \to (x^1, ..., x^m), m < n$$

A smooth curve $\gamma : J \to M$ is an immersion if and only if $\gamma'(t) \neq 0$ for each $t \in J$. The inclusion $\mathbf{S}^n \to \mathbf{R}^{n+1}$ is an immersion, where both manifolds are endowed with the standard smooth structures. Given smooth manifolds $M_1, ..., M_k$, the projections

$$M_1 \times ... \times M_k \to M_i$$

are submersions.

Let $J = (-\frac{\pi}{2}, \frac{3\pi}{2})$. Consider the curve

$$\gamma : J \to \mathbf{R}^2 \qquad \gamma(t) = (sin(2t), cos(t))$$

This curve is an injective immersion yet it is not an embedding.

Exercise 1.2. *Show that the curve in the last example is not an embedding.*

Now we recall two important theorems from multivariable calculus.

Theorem 1.3. *(Inverse function theorem) Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable. If the Jacobian matrix $dF : T_p(\mathbf{R}^n) \to T_{F(p)}(\mathbf{R}^n)$ is invertible at a point $p \in \mathbf{R}^n$, then F is locally a diffeomorphism, i.e. there is a neighbourhood U of p such that $F \restriction_U$ is a diffeomorphism onto its image.*

Theorem 1.4. *(Constant rank theorem) Let M, N be smooth manifolds of dimension m, n respectively. Let $F : M \to N$ be of constant rank in a neighbourhood of $p \in M$. Then there exist charts (U, ϕ) centred at p and (V, χ) centred at $F(p)$ such that the coordinate representation $\chi \circ F \circ \phi^{-1}(x^1, ..., x^m) = (x^1, ..., x^k, 0, ..., 0)$.*

This provides a "standard form" for any immersion or submersion.

1.1. Embedded submanifolds.

Definition 1.5. Let M be a smooth n-manifold. A subset $N \subset M$ is called an *embedded submanifold of dimension* k of M if for all $p \in N$ there exists a *slice chart*: A smooth chart (U, ϕ) with $p \in U$ such that

$$\phi(U \cap N) = \{x \in \phi(U) \mid x^j = 0 \text{ for } k + 1 \leq j \leq n\}$$

Example 1.6. Let $f : U \to \mathbf{R}^m$ be a smooth map where $U \subset \mathbf{R}^n$ is an open set. Then the graph

$$\Gamma(f) = \{(x, f(x)) \mid x \in U\} \subset \mathbf{R}^n \times \mathbf{R}^m$$

is an embedded submanifold in $\mathbf{R}^n \times \mathbf{R}^m$. The slice chart is

$$\phi : U \times \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^m \qquad (x, y) \in \Gamma(f) \qquad (x, y) \to (x, y - f(x))$$

Proposition 1.7. *Let M be a smooth manifold and let $N \subset M$ be a subset of M. If for all $p \in N$ there exists a neighbourhood U of p in M such that $U \cap M$ is an embedded submanifold of dimension k, then N is an embedded submanifold of dimension k.*

The proof of the above is an easy exercise.

Proposition 1.8. *Let $N \subset M$ be an embedded submanifold. of dimension k Then N has a unique smooth manifold structure such that the inclusion map $N \to M$ is a smooth embedding.*

Proof. We endow N with the subspace topology. Since subspaces of Hausdorff and second countable spaces have the same property, N has these features. For each $p \in N$, there is an open neighbourhood U of p in M and a slice chart (U, ϕ) such that

$$\phi(U \cap N) \subset \mathbf{R}^k \subset \mathbf{R}^n$$

Let $V = U \cap N$ and let $\pi : \mathbf{R}^n \to \mathbf{R}^k$ be the projection. Then (V, χ) for $\chi = \pi \circ \phi$ is a chart for N and V is homeomorphic to an open subset of $\mathbf{R}^k$. So for each slice chart (U_α, ϕ_α) of N (for α in some set I) we have a chart (V_α, χ_α) of N Since slice charts cover N, these charts make N locally Euclidean. This makes N a topological manifold.

Now we will show that these charts are smoothly compatible. Let (V_α, χ_α) and (V_β, χ_β) be charts. Then

$$\chi_\alpha \circ \chi_\beta^{-1} = (\pi \circ \phi_\alpha) \circ (\pi \circ \phi_\beta)^{-1} = \pi \circ (\phi_\alpha \circ \phi_\beta^{-1}) \circ \pi^{-1}$$

is smooth since it is a composition of smooth maps. Note that π^{-1} is simply the inclusion $\mathbf{R}^k \to \mathbf{R}^n$. This makes N a smooth manifold with the aforementioned system of charts.

Now we will show that this inclusion is a smooth embedding, i.e. an immersion which is a homeomorphism onto its image. Given a slice chart (U_α, ϕ_α) and the corresponding chart (V_α, χ_α) around $p \in M$, the inclusion $i : N \to M$ and the induced map $i_* : T_p(N) \to T_p(M)$ has the coordinate representation $\phi_\alpha \circ i \circ \chi_\alpha^{-1}$ which simply equals the inclusion $\mathbf{R}^k \to \mathbf{R}^n$. This is a map of constant rank $k < n$, hence an immersion. It is already clear that N is a homeomorphism onto its image. This finishes the proof.

Finally, we will show that this smooth structure on N is the unique smooth structure with the property that the inclusion is a smooth embedding. Let $\mathcal{A}'$ be another smooth structure on N which makes the inclusion an embedding. Let $(W, \eta) \in \mathcal{A}'$, and let (V_α, χ_α) be a smooth chart in the aforementioned smooth structure on N. We simply need to show that the map

$$\chi_\alpha \circ i \circ \eta^{-1} : \eta(W \cap V_\alpha) \to \chi_\alpha(W \cap V_\alpha)$$

is a diffeomorphism to show that in fact (W, η) belongs in the original set of charts. We will show that the Jacobian at each point is regular, and by the inverse function theorem this will imply that this map is a diffeomorphism. Note that

$$d(\chi_\alpha \circ i \circ \eta^{-1}) = d\pi \circ d\phi_\alpha \circ di \circ d\eta^{-1}$$

$di \circ d\eta^{-1}$ is of full rank since (W, η) is a chart in an atlas that makes $N \to M$ and embedding. And $d\pi \circ d\phi_\alpha$ is of rank k, since $d\phi_\alpha$ is surjective on the first k coordinates. This makes the composition regular. □

Corollary 1.9. *The standard smooth structure on $\mathbf{S}^n$ is the unique smooth structure that makes it an embedded submanifold of $\mathbf{R}^n$.*

Proposition 1.10. *Let M, N be smooth manifolds of dimenison n, k and let $F : N \to M$ be a smooth embedding. Then the image $F(N)$ is an embedded submanifold of M of dimension k.*

Proof. We need to construct slice charts. Let $p \in F(N)$. By the constant rank theorem, we can find a smooth chart (U, ϕ) around $F(p) \in M$, and a smooth chart $(V = F^{-1}(U \cap F(N)), \chi)$ of N such that the coordinate representation $\phi \circ F \circ \chi^{-1}$ is of the form

$$(x^1, ..., x^k) \to (x^1, ..., x^k, 0, ..., 0)$$

So the required slice chart is (U, ϕ). □

Corollary 1.11. *Embedded submanifolds are precisely the images of smooth embeddings.*

The following is a famous result in differential geometry.

Theorem 1.12. *(Whitney embedding theorem) Any smooth n-manifold can be embedded in $\mathbf{R}^{2n}$.*

Embedded submanifolds often arise as level sets of smooth maps of constant rank.

Theorem 1.13. *(Constant rank level set theorem) Let $F : M \to N$ be a smooth map of constant rank k. Then the level set $F^{-1}(p)$ for $p \in N$ is an embedded submanifold of codimension k in M.*

Proof. Let $S = F^{-1}(p)$ and let $q \in S$. We shall construct a slice chart for S in M at q. By the constant rank theorem, there exist charts (U, ϕ) centred at q and (V, χ) centred at $F(q) = p$, such that

$$\chi \circ F \circ \phi^{-1}(x^1, ..., x^n) = (x^1, ..., x^k, 0, ..., 0)$$

with $\chi(p) = 0$ and $\phi(q) = 0$. This means that

$$\phi(U \cap S) = \{x \in \phi(U) \mid x^j = 0 \mid 1 \leq i \leq k\}$$

This means that the chart (U, ϕ) is a slice chart for S. □

Now we shall prove that any embedded submanifold is, locally, given as the level set of a submersion.

Proposition 1.14. *Let S be a subset of a smooth manifold M. Then S is an embedded submanifold of dimension k if and only if each point $p \in M$ admits an open neighbourhood U such that $U \cap S$ is the level set of a smooth submersion $F : M \to \mathbf{R}^{n-k}$.*

Proof. One direction follows from Theorem 1.13. Now let S be an embedded submanifold of M, let $p \in S$ and let (U, ϕ) be a slice chart centred at p. Then

$$\phi(U \cap S) = \{x \in \phi(U) \mid x^j = 0, k+1 \leq j \leq n\}$$

Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$ be the projection onto the last $n-k$ coordinates. Then the required submersion is $\pi \circ \phi : U \to \mathbf{R}^{n-k}$. □

Definition 1.15. Let $F : M \to N$ be a smooth map. We say that $p \in M$ is a *regular point* of F if $F_* : T_p(M) \to T_{F(p)}(N)$ is surjective, and is said to be a *critical point* otherwise. A point $q \in N$ is said to be a *regular value* if each point $p \in F^{-1}(q)$ is a regular point of F. In this case, $F^{-1}(q)$ is called a *regular level set*.

Proposition 1.16. *(Regular level set theorem) Every regular level set of a smooth map $F : M \to N$ is an embedded submanifold whose codimension equals the dimension of N.*

Exercise 1.17. *Prove the above proposition.*

Example 1.18. We can describe $\mathbf{S}^{n-1}$ as a level set of the map

$$\mathbf{R}^n \to \mathbf{R} \qquad (x^1, ..., x^n) \mapsto \sum_{1 \leq i \leq n} (x^i)^2 \qquad \mathbf{S}^n = F^{-1}(\{1\})$$

Example 1.19. The special linear group $SL_n(\mathbf{R})$ is the group of $n \times n$ matrices with determinant equal to 1. It is a level set of the determinant function, $Det : GL_n(\mathbf{R}) \to \mathbf{R}$. For $A \in GL_n(\mathbf{R})$, we identify $T_A(GL_n(\mathbf{R})) \cong M(n)$ and as usual $T_{Det(A)}(\mathbf{R}) \cong \mathbf{R}$. The pushforward is given by

$$det_* \restriction_A (B) = (Det(A)) tr(A^{-1}B)$$

Since this is always rank 1 surjective, this means that $SL_n(\mathbf{R})$ is an embedded submanifold of $GL_n(\mathbf{R})$.

Definition 1.20. If $S \subset M$ is an embedded submanifold and if $U \subset M$ is open, $F : U \to N$ is smooth such that $U \cap S$ is a regular level set of F, then F is said to be a local defining map for S.

Proposition 1.21. *Suppose that $S \subset M$ is an embedded submanifold and for $U \subset M$ an open set we have $F : U \to N$ a local defining map for S. Then*

$$i_*(T_p(S)) = ker(F_*) \mid_p$$

for each $p \in S \cap U$ and $i : S \to M$ the inclusion map.

Proof. Suppose that $S \cap U = F^{-1}(q)$ for $q \in N$. Let the dimension of M be n and the dimension of N be k.

First we will show that $i_*(T_p(S)) \subseteq ker(F_*) \mid_p$. Note that $F \circ i \restriction S$ is constant, so it follows that $(F \circ i)_* = F_* \circ i_* \restriction T_p(S) = F_* \restriction i_*(T_p(S))$ is trivial. So we conclude that $i_*(T_p(S)) \subseteq ker(F_*) \mid_p$.

We conclude by showing that the dimensions of $i_*(T_p(S)), ker(F_*) \mid_p$ are equal. Note that $dim(i_*(T_p(S))) = n-k$ and since F_* is surjective we have that $dimker(F_*) \mid_p$ is also $n-k$.

□