LECTURE 8

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1. Differential forms

Integrating a smooth function $f : M \to \mathbf{R}$ poses some problems, namely if we do so fixing a coordinate chart locally, then it is tricky to patch together this local integral into a global one. In particular, this appears to depend on the choice of local coordinate charts. A sensible solution to this problem is the notion of *differential forms*, which we shall be studying now.

1.1. The cotangent space.

Definition 1.1. Let M be a smooth manifold and let $p \in M$. The *cotangent space* $T_p^*M = (T_pM)^*$ is the dual space of the tangent space T_pM. Its elements are called *cotangent vectors* consist of linear maps $T_pM \to \mathbf{R}$.

Let (U, ϕ) be a coordinate chart centred at p. Recall that the basis of T_pM with respect to this chart is $\frac{\partial}{\partial \phi^1}, ..., \frac{\partial}{\partial \phi^n}$. For the moment, we denote the basis of the dual space T_p^*M as $\varepsilon_p^1, ..., \varepsilon_p^n$ where

$$\varepsilon_p^i(\frac{\partial}{\partial \phi^j}) = \delta_j^i$$

Here $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ if $i \neq j$.

Definition 1.2. Let $f : U \to \mathbf{R}$ be a smooth function defined on an open subset $U \subset M$. Then the differential $df_p : T_pM \to T_{f(p)}\mathbf{R} \cong \mathbf{R}$ is a cotangent vector in T_p^*M. Now let (U, ϕ) be a chart centred at p. Consider the coordinate function $\phi^i : U \to \mathbf{R}$. The differential is

$$d\phi_p^i : T_pM \to T_{\phi^i(p)}\mathbf{R} \cong \mathbf{R} \qquad d\phi_p^i(\frac{\partial}{\partial \phi^j}) = \frac{\partial}{\partial \phi^j} \restriction_p \phi^i = \delta_j^i$$

This means that $d\phi_p^i = \varepsilon_p^i$. Therefore, the coordinate chart provides the basis $d\phi_p^1, ..., d\phi_p^n$ for the cotangent space.

For an arbitrary smooth function we have

$$df_p : T_pM \to T_{f(p)}\mathbf{R} \qquad df_p(\frac{\partial}{\partial \phi^i}) = \frac{\partial}{\partial \phi^i} \restriction_p f$$

So we obtain

$$df_p = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial \phi^i} \restriction_p f) d\phi^i$$

Definition 1.3. (Notational convention) To simplify notation, for the rest of this section we shall denote $d\phi_p^i$ as dx_p^i and $\frac{\partial}{\partial \phi^i} \restriction_p$ as $\frac{\partial}{\partial x^i} \restriction_p$. The map ϕ shall be implicit in our assumptions. So

$$df_p = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial x^i} f(p)) dx_p^i$$

If $\tilde{x}^1, ..., \tilde{x}^n$ is another chart centred at p, then the above provides

$$d\tilde{x}_p^j = \sum_{1 \leq i \leq n} \frac{\partial \tilde{x}^j}{\partial x^i}(p) dx_p^i \qquad 1 \leq j \leq n$$

1.2. **The cotangent bundle and differential 1-forms.** The set $T^*M = \bigcup_{p\in M} T_p^*M$ is called the *cotangent bundle* of M.

Proposition 1.4. *The cotangent bundle has a natural smooth structure making it a vector bundle of rank n over M with projection $\pi : T^*M \to M$ sending $\psi \in T_p^*M$ to p.*

Proof. The natural smooth structure is described in the following way. Let (U, ϕ) be a chart which contains $p \in M$. Then

$$\Phi : \pi^{-1}(U) \to \mathbf{R}^{2n} \qquad \Phi(\sum_{1\leq i\leq n} \zeta^i dx_p^i) = (\phi(p), \zeta) \qquad \zeta = (\zeta^i)_{1\leq i\leq n}$$

The topology is defined in an analogous manner as in the tangent bundle, and the rest of the proof is similar. □

Definition 1.5. A smooth section of T^*M is called a *differential* 1*-form*, or simply, a 1-form. The set of 1-forms on M is denoted as $\Omega^1(M)$. Clearly, $\Omega^1(M)$ is closed under pointwise addition and multiplication by elements of $C^\infty(M)$.

Given $\sigma \in \Omega^1(M)$, we write $\sigma_p = \sigma(p)$. Given a chart (U, ϕ), recall that $dx_p^1, ..., dx_p^n$ is a basis for $T_p^*(M)$. We write

$$\sigma_p = \sum_{1\leq i\leq n} \sigma_i(p) dx_p^i$$

One natural way to obtain covector fields is as differentials of smooth functions $f : M \to \mathbf{R}$.

$$df : M \to T^*M \qquad df(p) = df_p$$

A 1-form $\sigma \in \Omega^1(M)$ is said to be *exact* if there exists $f \in C^\infty(M)$ such that $\sigma = df$.

The following are some elementary properties of the differential.

Proposition 1.6.

(1)
$$d : C^\infty(M) \to \Omega^1(M)$$
is a linear map.

(2) *If $h : \mathbf{R} \to \mathbf{R}$ is smooth and $f \in C^\infty(M)$, then $d(h \circ f) = (h' \circ f)df$.*

(3) *If $df = 0$ then f is constant on each connected component of M.*

Exercise 1.7. *Prove the above.*

1.3. **Pullbacks.**

Definition 1.8. Let $F : M \to N$ be a smooth map between manifolds. Recall that the pushforward is defined as

$$F_* : T_pM \to T_{F(p)}M$$

The *pullback* is a map

$$F^* : T^*_{F(p)}N \to T_p^*M \qquad (F^*\psi)(\tau) = \psi(F_*(\tau)) \qquad \psi \in T^*_{F(p)}N, \tau \in T_pM$$

The pullback satisfies that

$$(F \circ G)^* = G^* \circ F^*$$

One can always "pullback" covector fields as follows:

$$F^*(\sigma)_p = F^*(\sigma_{F(p)}) \qquad \sigma \in \Omega(N)$$

(Note that in general is may not be possible to "pushforward" a vector field. Can you see why?) We will soon show that the pullback is also a smooth covector field.

Lemma 1.9. *Let $F : M \to N$ be a smooth map between smooth manifolds, $f \in C^\infty(N), \sigma \in \Omega^1(N)$. Then*

$$F^*df = d(f \circ F) \qquad F^*(f\sigma) = (f \circ F)F^*\sigma$$

Proposition 1.10. *Let $F : M \to N$ be a smooth map between smooth manifolds, $\sigma \in \Omega^1(N)$. Then the pullback $F^*\sigma$ is a smooth covector field on M.*

Exercise 1.11. *Prove the above.*