Quiz Solution 8 Discussed on Zoom

① $$\frac{1}{M} \operatorname{Var}_{\psi}\left(f(x)\, w(x) \right) = V \geq 0$$

$$w(x) = \frac{\pi(x)}{\psi(x)}$$

$$V = \frac{1}{M}\left\{ \sum_x f(x)^2 w(x)^2 \psi(x) - \left(\sum_x f(x) w(x) \psi(x) \right)^2 \right.$$

$$= \frac{1}{M}\left\{ \sum_x \frac{f(x)^2 \pi(x)^2}{\psi(x)} - \underbrace{\left(\sum_x f(x)\pi(x) \right)^2}_{\left(\mathbb{E}_{\pi}(f(x))\right)^2} \right\}$$

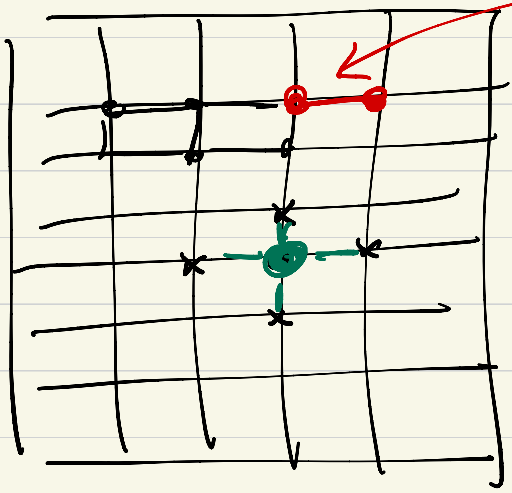
$\geq 0.$

↑ is this attained for some ψ? YES.

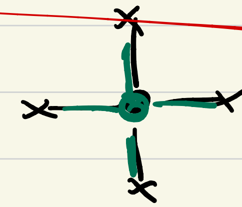
$$\boxed{\text{TAKE} \quad \psi(x) = \frac{f(x)\pi(x)}{\sum f(x)\pi(x)}} \Rightarrow \boxed{V = 0}$$

What is the problem in practice? $\sum_x f(x)\pi(x)$. ?

(2)



For this edge
$J_{kl} S_k S_l = J_{kl} S'_k S'_l$.

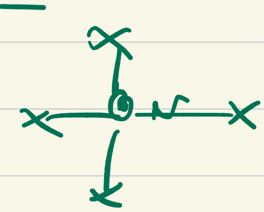
v 

only here do we flip the spin.

$$\frac{\pi(S')}{\pi(S)} = \frac{e^{-\beta \sum_{k,l \in E} J_{kl} S'_k S'_l}}{e^{-\beta \sum_{k,l \in E} J_{kl} S_k S_l}}$$

$$= \exp -\beta \left[\sum_{k,l \in E} J_{kl} S'_k S'_l - \sum_{k,l \in E} J_{kl} S_k S_l \right].$$

$\underline{S}$ and $\underline{S'}$ differ only at vertex $\underline{v}$.

Lots of simplification except for edges 

$$= \exp -\beta \left[2 \underbrace{\sum_{k \text{ neib to } v}}_{\substack{4 \text{ terms} \\ 2d \text{ terms}}} J_{kv} S_k \overset{-S_v}{S'_v} - 2 \underbrace{\sum_{k \text{ neib to } v}}_{\substack{4 \text{ terms} \\ 2d \text{ terms.}}} J_{kv} S_k S_v \right]$$

$\boxed{O(d) \text{ terms to compute}}$

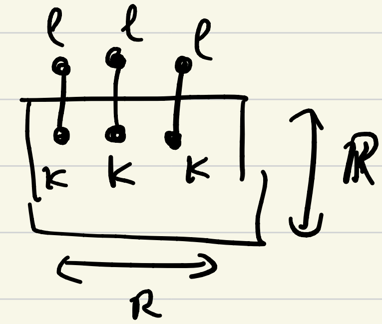
3

Square Grid:

- If an edge is outside the red region again $s'_k s'_\ell = s_k s_\ell \to$ terms cancel.

- If an edge is inside the red region then again $s'_k s'_\ell = \underbrace{(-s_k)(-s_\ell)}_{\text{two flips}} = s_k s_\ell$ $\to$ terms cancel

- If an edge is



we have $s'_k s'_\ell = (-s_k) s_\ell = -s_k s_\ell$

$\to$ terms do Not cancel.

$\Rightarrow$ # of terms in ratio $\frac{\pi(S')}{\pi(S)}$ is $O(R)$ perimeter ✓

Tree:

On a tree by the same argument we have to count the # of terms at leaves of red region.

This is $\sim \Delta^R$ and for $R = \alpha \log N$ we find

(Δ = degree of vertices) $\Delta^R = \Delta^{\alpha \log N} = e^{(\alpha \log \Delta) \log N}$

$= N^{\alpha \log \Delta}$ poly in N