Homework 10

Exercise 1. [Metropolized independent sampling in a particular case]
Let $0 < \theta < 1$ and let us consider the following distribution π on $S = \{1, \dots, N\}$:

$$\pi_i = \frac{1}{Z}\,\theta^{i-1}, \quad i = 1, \dots, N$$

where Z is the normalization constant, whose computation is left to the reader.

a) Consider the base chain $\psi_{ij} = \frac{1}{N}$ for all $i, j \in S$ and derive the transition probabilities p_{ij} obtained with the Metropolis-Hastings algorithm.

b) Using the result of the course, derive an upper bound on $\|P_i^n - \pi\|_{\mathrm{TV}}$. Compare the bounds obtained for $i = 1$ and $i = N$ (for large values of N).

c) Deduce an upper bound on the (order of magnitude of the) mixing time

$$T_\varepsilon = \inf\{n \geq 1 \,:\, \max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \varepsilon\}$$

Exercise 2. [Sampling from a posterior]
Assume you are given some "data points" $y_1, \dots, y_N$ which are known to be i.i.d samples of $q(y \mid \theta)$ where $\theta \in \mathbb{R}$ is a random parameter with (continuous) prior $p_0(\theta)$.

We want to sample θ from the posterior distribution $p(\theta \mid y_1, \cdots, y_N)$. We decide to construct a MH Markov chain $\theta^0, \theta^1, \theta^2, \dots, \theta^t, \theta^{t+1}, \dots$ where the proposal move consists of the transition $\theta^t \to \theta^{t+1}$ with probability $p_0(\theta^{t+1})$.

a) Formulate the MH algorithm and in particular give a *simple* formula for the acceptance probabilities that does not involve the prior and/or any potentially difficult to compute integral.

b) Is it true that you do not need to know this prior in order to run the chain ? What is the advantage of MH here w.r.t directly sampling the posterior ?